Complete — with index

THE MATHEMATICAL GAZETTE

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.

AND

PROF. E. T. WHITTAKER, M.A., F.R.S.

LONDON

G. BELL & SONS, LTD., PORTUGAL STREET, KINGSWAY, W.C. 2.

AND BOMBAY

Vol. XI., No. 156. JANUARY, 1922. 2s. 6d. Net.

The Mathematical Gazette is issued in January, March, May, July, October, and December.

All correspondence concerning the *contents* of the **Gazette** should be addressed to W. J. GREENSTREET, The Woodlands, Burghfield Common, nr. Mortimer, Berks.

Correspondence relative to the **Mathematical Association** or the *distribution* of the **Gazette** should be addressed to one of the Hon. Secretaries:—C. PENDLEBURY, 39 Burlington Road, Chiswick, W. 4; Miss M. PUNNETT, B.A., London Day Training College, Southampton Row, W.C. 1.

Change of address should be notified to a Secretary.

If Copies of the Gazette fail to reach a member in consequence of a change of address, of which the Secretaries have not been informed, duplicate copies cannot be supplied.

CONTENTS.

NOTICE.

HON. SECRETARIES OF BRANCHES OF THE MATHEMATICAL ASSOCIATION.

LONDON:	Miss L. A. ZELENSKY, Haberdashers' Aske's School, Acton, W. 3.
	Mr. G. GOODWILL, 37 Park Road, Dulwich, S.E. 21.
SOUTHAMPTON:	(Office temporarily vacant.)
NORTH WALES:	Mr. T. G. CREAK, Ty Mawr, Clwtybont, Cwm-y-glo, S.O., North Wales.
SYDNEY (N.S.W.):	Mr. H. J. MELDRUM, Teachers' College, Sydney.
	Miss F. COHEN, Girls' High School, Sydney.
YORKSHIRE (Leeds):	Rev. A. V. BILLEN, The Grammar School, Leeds.
BRISTOL:	Mr. R. C. FAWDRY, Clifton College.
MANCHESTER:	Miss W. GARNER, Whalley Range High School.
	Miss M. L. TANNER, Broughton High School.

THE
MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.
WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., AND PROF. E. T. WHITTAKER, M.A., F.R.S.

LONDON:
G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY,
AND BOMBAY.

VOL. XI. JANUARY, 1922. No. 156.

STATISTICS AS APPLIED TO EDUCATIONAL QUESTIONS.*

BY H. J. MELDRUM, B.A., B.Sc.

IT is becoming more widely recognised each year, among leaders of educational thought, that in the field of educational science, where mere opinion holds sway, this opinion should be replaced by conclusions scientifically obtained from carefully collected data.

In the teaching of mathematics we have such questions as:

(i) What is the most economical length of drill periods in first-year algebra ?

(ii) What is the best method of procedure for acquiring skill in the mechanical work in algebra ?

(iii) To what extent does mechanical skill in algebra help the pupil intelligently to use algebraic processes, such as the equation, the graph and the formula ?

(iv) What proportion of pupils benefits by a certain type of course and up to a given standard ?

(v) The results of (iv) would be a factor in deciding whether all secondary school pupils should take mathematics throughout their course.

We have then questions of vital interest (*a*) to class teachers of mathematics, (*b*) to those responsible for the designing of curricula, etc.

None of these questions can be settled easily or off-hand; but many of them can and will be solved by the application of the theory of statistics. It is hardly necessary to point out that no solution that is found by such application of scientific mathematical methods can be regarded as final. The history of physical science illustrates how tentative solutions are often modified, sometimes radically, even though in the first place the result was the work of much careful observation and experiment. If that is true of physical material, the conditions of which can be regulated with comparative ease and certainty, it will hold with greater force when we are dealing with such complex beings as school pupils.

One of the important conceptions in the Theory of Statistics is that of correlation, and I will now try to indicate the significance of this "correlation co-efficient," as it is called. A measure of the degree of association between two sets of results can be made in different ways. We will confine our attention to one—that which is in most common use and known as the Bravais-Pearson

* (Read at a meeting of the Sydney Branch of the Mathematical Association, 21st May, 1920.)

coefficient. It will help to fix our ideas if we refer to the Table following, where the numbers X and Y are the marks obtained by pupils in two tests; thus, the pupil No. 1 obtained 40 marks in each test; pupil No. 2 obtained 37 marks and 43 marks in the respective tests; and so on. (The other columns will be explained later.)

Pupil.	X	Y	x +	x −	y +	y −	x^2	y^2	xy +	xy −
1.	40	40		3·3		·3	10·89	·09	·99	
2.	37	43		6·3	2·7		39·79	7·29		17·01
3.	53	46	9·7		5·7		94·09	32·49	55·29	
4.	37	30		6·3		10·3	39·79	106·09	64·89	
5.	61	55	17·7		14·7		313·29	216·09	260·19	
6.	34	30		9·3		10·3	86·49	106·09	95·79	
7.	68	65	24·7		24·7		610·09	610·09	610·09	
8.	41	43		2·3	2·7		5·29	7·29		6·21
9.	41	43		2·3	2·7		5·29	7·29		6·21
10.	33	34		10·3		6·3	106·09	39·79	64·89	
11.	45	35	1·7			5·3	2·89	28·09		9·01
12.	62	52	18·7		11·7		349·69	136·89	218·79	
13.	52	53	8·7		12·7		75·69	161·29	110·49	
14.	45	45	1·7		4·7		2·89	22·09	7·99	
15.	41	46		2·3	5·7		5·29	32·49		13·11
16.	30	27		13·3		13·3	176·89	176·89	176·89	
17.	39	31		4·3		9·3	18·49	86·49	39·99	
18.	41	32		2·3		8·3	5·29	68·89	19·09	
19.	40	27		3·3		13·3	10·89	176·89	43·89	
20.	34	35		9·3		5·3	86·49	28·09	49·29	
21.	44	42	·7		1·7		·49	2·89	1·19	
22.	48	44	4·7		3·7		22·09	13·69	17·39	
23.	57	59	13·7		18·7		187·69	349·69	256·19	
24.	44	32	·7			8·3	·49	68·89		5·81
25.	26	30		17·3		10·3	299·29	106·09	178·19	
26.	30	28		13·3		12·3	176·89	151·29	163·59	
27.	54	48	10·7		7·7		114·49	59·29	82·39	
28.	19	17		24·3		23·3	590·49	542·89	566·19	
29.	53	51	9·7		10·7		94·09	114·49	103·79	
30.	55	57	11·7		16·7		136·89	278·89	195·39	
31.	39	33		4·3		7·3	18·49	53·29	31·39	
32.	41	35		2·3		5·3	5·29	28·09	12·19	
	M= 43·3	M= 40·3								

$$\sigma_x=\sqrt{\frac{\Sigma x^2}{N}}=\sqrt{\frac{3692{\cdot}28}{32}}=10{\cdot}7\,; \qquad \Sigma(xy)=3426{\cdot}44-57{\cdot}36$$

$$\sigma_y=\sqrt{\frac{\Sigma y^2}{N}}=\sqrt{\frac{3820{\cdot}18}{32}}=10{\cdot}9\,; \qquad =3369{\cdot}08$$

$$r=\frac{3369{\cdot}08}{32\times 10{\cdot}7\times 10{\cdot}9}=\cdot 9.$$

The relationship we want to show can be illustrated graphically. We may group the marks, taking those having 0-4 in one group, 5-9 in next group, 10-14 in the next, and so on. (This class interval can of course be made larger or smaller.) If we look at the square corresponding to 40—along the X-axis and 40—along the Y-axis, the four dots indicate that four of the candidates

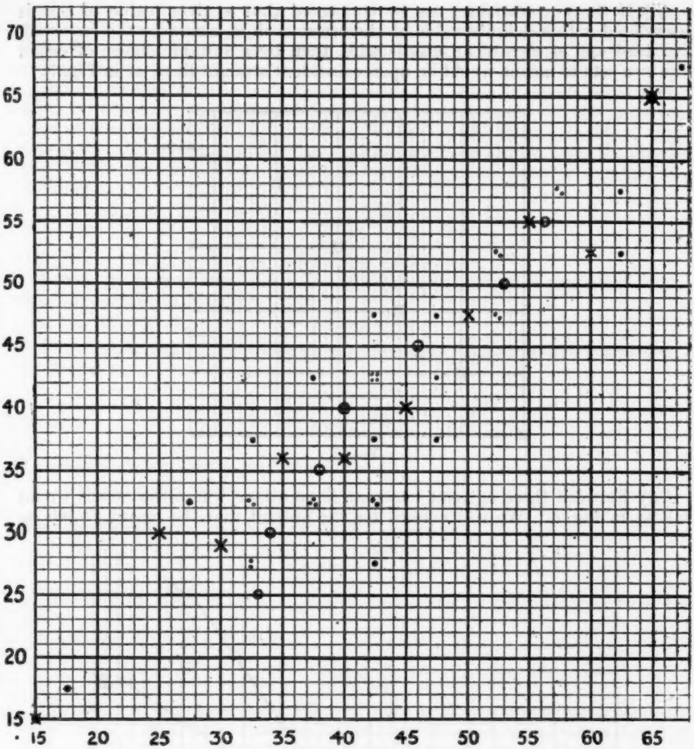


Fig. 1.

obtained marks in the group 40-44 in each test. The means of the marks for " rows " and " columns " are then marked respectively by circles and crosses. Although these means are in this case not in straight lines, we can " fit " straight lines for them. This is done by considering the following theoretical cases :

(i) If the two quantities X and Y are independent, then if a large number of measurements are made, these measurements will be scattered uniformly over the whole field. The means of rows and columns would then be on straight lines at right angles, as in Fig. 2.

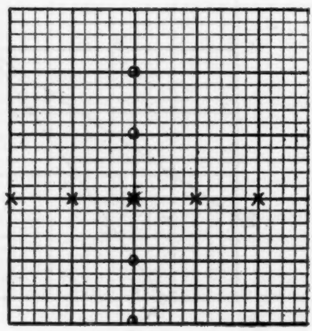


Fig. 2.

(ii) If the two variables are so connected that a given value of one is associated with a definite value of the other, such as would be the case with the load and stretch of an elastic wire, then by a suitable choice of origin the two sets of means would lie upon coincident straight lines, as in Fig. 3.

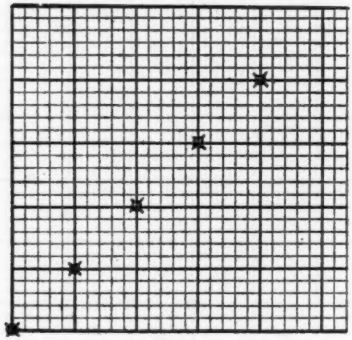


FIG. 3.

(iii) Between these extreme cases we have others where the means would fall as in Fig. 4. We will assume for the present that the means *do* fall on

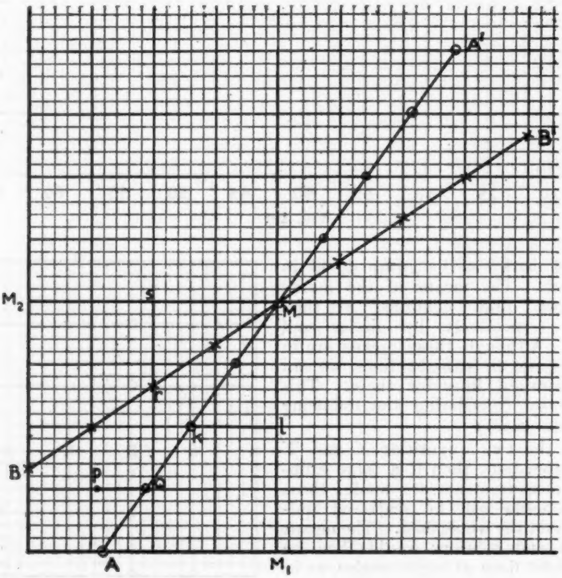


FIG. 4.

straight lines. It can be shown that the "lines of regression" AA', BB' pass through M, where M_1, M_2 are the means of X and Y respectively. Taking M as the origin, then for any point k, $kl\ (=x)$ is the deviation of this mean-measure from the mean of all measures X.

Before dealing further with the properties of these lines, we must define what is meant by the *Standard Deviation*. Some measure of the scatter of a set of measurements is often needed; the S.D. is the most valuable of these measures. It is defined as the "root-mean-square-deviation from the mean." If x is the deviation of any measure from the mean, then in symbols the S.D. is

$$\sigma=\sqrt{\frac{\Sigma x^2}{N}},$$

where N is the total number of measures taken.

Let us go back to Fig. 4. In the straight line AA', take any point k, and draw kl parallel to X-axis.

If $\tan M'MA=b_1$, then $\qquad x=b_1 y.$

But x is the mean of all measures ξ in that row; therefore, as y is a constant for this row,

$$\Sigma(\xi y)=y\Sigma\xi=nb_1y^2,$$

since $\qquad x=\bar{\xi}=\frac{\Sigma\xi}{n},$

where there are n measures in this row.

$\therefore$ for the whole table,

$$\Sigma(\xi y)=b_1\Sigma(ny^2)=b_1N\sigma_y^2$$

from the definition of the S.D. of the Y-measures,

$$i.e.\quad b_1=\frac{\Sigma(xy)}{N\sigma_y^2},$$

where we may write x for ξ to denote deviations of the measures from the mean.

Similarly, if $b_2=\tan M_2MB$,

$$b_2=\frac{\Sigma xy}{N\sigma_x^2}.$$

The slope of the lines AA', BB' will vary with the degree of dependence of X and Y; so that a measure of this dependence can be made a function of b_1 and b_2. We may therefore define this correlation of the two sets of measurements as the geometric mean of b_1 and b_2, or

$$r=\sqrt{b_1 . b_2}$$
$$=\frac{\Sigma xy}{N\sigma_x . \sigma_y}.$$

This coefficient has been worked out for the table above.

In any actual case, the means of rows and columns will not lie on straight lines. But it can be shown that the sum of the squares of the deviations of measures from these lines is a minimum in each case.

Referring to Fig. 4 for the X-measures, if P is the actual position of one of the measures, PQ is the deviation of this measure from the line AA'. It is the sum of all measures, such as PQ^2, that is referred to.

(One proof will be found in Yule, *An Introduction to the Theory of Statistics*, Chap. IX. The above discussion of lines of regression follows the treatment in that chapter.)

We can now follow the purpose of Table I. more readily. The results were taken from among those that arose in connection with an investigation into the question of the methods of teaching subtraction. The two methods of performing a simple subtraction under review are known as the "Equal

Additions" and "Decomposition" methods respectively. We need not discuss this question further, except to point out that test papers were compiled in each of the four fundamental operations. When these tests have been given, one question to consider is the "reliability" of the results; that is, whether on the whole the results satisfactorily represent relative abilities of the pupils in the operations considered. It sometimes happens, through a variety of causes, that the results of a test do not represent the abilities of the pupils satisfactorily. The reliability of the test is measured by making use of the correlation coefficient in the following way: the test is repeated in the conditions of the first trial, and the correlation coefficient between the two sets of results is found. If this coefficient is high, certainly not less than 0·6, then it is highly probable that the results obtained are on the whole satisfactory, from the point of view mentioned.

The results in the Table are those of a particular class for the two tests in addition; and the value of r (·9) shows that the results are quite reliable.

In this investigation the reliability of all tests in all classes was found before any deductions were made from these results.

Theory of Errors. In the investigation of the two methods of subtraction, after it had been determined that the tests were reliable, the means were found for each operation, both for marks gained and times taken. The corresponding means were then compared for the pupils using the Equal Additions method with those using the Decomposition method. This raises the question as to the significance of the differences of the means, which introduces the Theory of Errors, or, as a particular case, the Theory of Sampling.

If we suppose that there were 400 pupils tested for each method of subtraction, even if the results for the pupils tested were reliable, we would most likely get a slightly different set of means if 400 different pupils were chosen. We would place more reliance on the results of 4000 pupils, and the most reliance on the results of the tests if *all* pupils could be tested. The 400 would be a "sample"; and it is assumed that it will depend on chance whether the means obtained from any such sample are above or below the mean that would be obtained from all pupils—the latter being the "true" mean.

On this assumption, it is argued that for a large number of samples the means will be symmetrically ranged about the "true" mean, being most numerous near it, and less frequent further from it. This is also in accordance with observation.

If we take the position of the hypothetical true mean as origin, deviations of the observed means from this true mean for the x-variable, and the frequency of the observed means for the y-variable, then it can be shown that the points so fixed would tend to the curve

$$y = y_0 e^{-\frac{x^2}{2\sigma^2}}.$$

This is the so-called "normal curve of error."

[A proof will be found in Dr. Nunn's book, *Exercises in Algebra*, Part II., in the chapter on Statistics. Another proof is in Weld, *The Theory of Errors and Least Squares.*]

We have to consider what will be the probability that the mean of any sample will deviate from the true mean by any given amount.

Let Fig. 5 represent the curve of error, with O at the position of the true mean.

Take the points M_1, M_2, such that $OM_1 = OM_2$, and so placed that the ordinates through M_1 and M_2 enclose half the total area under the curve; the probability is $\frac{1}{2}$ that a given mean will fall within the range $M \pm q$, where

$$M = \text{true mean}, \quad q = OM_1 = OM_2.$$

This quantity q is known as the Probable Error of the mean. Its calculation from known quantities is discussed in books on Statistics.

Following up this line of argument, we have the Probable Error of the difference of two means. The usual assumption made is that, if the observed difference is not less than three times the Probable Error, then this difference would not be due to "errors of sampling"; that is, the difference is significant.

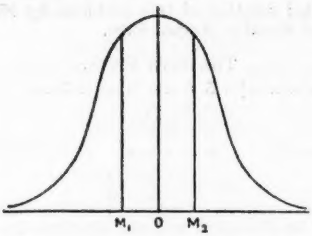


FIG. 5.

In the time at my disposal, it has been possible to do little more than indicate one or two types of problem that the study of statistics will help to solve; and, in doing so, I have followed well-beaten paths. While few may follow up the study of the Theory of Statistics, more may be able to use statistical methods in carrying out investigations; but it is becoming increasingly necessary for all teachers to be able to follow intelligently the results of investigations.

REFERENCES.

For the theory—G. V. Yule, *An Introduction to the Theory of Statistics.* Not the least valuable part of this book is the full list of references to original works.

The work of Dr. Nunn mentioned above is a valuable introduction to the study of the subject.

Several text-books have been written specially for teachers, dealing with educational problems mainly. These are not so much concerned with the theory as with the application to examples, and among them we may mention Thorndike, *Social and Mental Measurements.*

H. J. MELDRUM.

Teachers' College, Sydney.

GLEANINGS FAR AND NEAR

93. Mathematics, it is true, are recommended for this purpose, to fix the attention; but then the study of them is so tedious, and life is so short, and again, the truth they discover is altogether so absolute, and unrelated to the happiness of man, that I should content myself with a general knowledge of them.—John Dunton's "Idea of a New Life": p. 36, vol. i., *Life and Errors*, 1818 ed.

94. Mr. Mortimer, who came from Ireland, . . . was an accomplished Merchant, a person of great modesty, and could answer the most abstruse points in Algebra, Navigation, Dialling, etc.—Dunton's *Life and Errors*, i. 97.

95. I will begin with Mrs. Ab—l (a person of quality); a well-wisher to the mathematics; a young Proficient, but willing to learn, and therefore came to enquire for "The School of Venus."—Dunton's *Life and Errors*, i. 109.

SOLUTIONS TO MISSING-FIGURE PROBLEMS.

By W. E. H. Berwick.

The Seven Sevens.

125473) 7375428413 (58781.

An excellent detailed solution of this problem, by Mr. J. W. Reid, is to be found in the *School World* of August 1906.

The Five Fives.

$$
\begin{array}{rlr}
\times\times\times\times)\times 5\ 5 \times\times 5 \times(\times 5 \times & & (1)\\
\times\times 5 \times\times & & (2)\\
\times\times\times\times\times & & (3)\\
\times\times\times\times\times & & (4)\\
\times\times\times\times & & (5)\\
\times\times\times\times & & (6)
\end{array}
$$

(i) Let the divisor be $D \equiv \alpha\beta\gamma\delta$ and the quotient $Q \equiv \epsilon 5\theta$: from lines (4) and (6) θ cannot exceed 4. Since $5 . D \geqslant 10^4$ and $\theta . D < 10^4$, the following are the limits for D:

$$\theta=1,\ 2000 \leqslant D \leqslant 9999;\quad \theta=3,\ 2000 \leqslant D \leqslant 3333;$$
$$\theta=2,\ 2000 \leqslant D \leqslant 4999;\quad \theta=4,\ 2000 \leqslant D \leqslant 2499.$$

(ii) The limits for line (3), $uv \times \times 5$, are 10005-10995, 19015-20995, 29015-30995, 39015-40995, 49015-50995; and since this line $=5\theta . D$, the divisor is further restricted, viz.:

	$\theta=1$	$\theta=2$	$\theta=3$	$\theta=4$
$uv=10$,	2000-2156,	2000-2115,	2000-2075,	2000-2037.
$uv=19, 20$,	3726-4117,	3654-4038,	—	—
$uv=29, 30$,	5687-6078,	—	—	—
$uv=39, 40$,	7648-8039,	—	—	—
$uv=49, 50$,	9608-9999,	—	—	—

(iii) Since $5\theta . D$ ends with $5 \times$, from line (3), we have the following possibilities for $\gamma\delta$:

$\theta=1$, $\gamma\delta=$01, 03, 05, 07, 09, 50, 52, 54, 56, 58;
$\theta=2$, $\gamma\delta=$01, 03, 26, 28, 51, 53, 76, 78;
$\theta=3$, $\gamma\delta=$01, 03. 18, 35, 50, 52, 67, 69, 84, 86;
$\theta=4$, $\gamma\delta=$01, 14, 25, 27, 38, 51, 64, 75, 77, 88.

(iv) Let the dividend be $\lambda 55 \times \times 5 \times$, then, subject to (ii), the limits for the divisor are

$\theta=1$, $\lambda=3$, $\epsilon=4$, 7872-7893; $\lambda=2$, $\epsilon=6$, 3918-3832;
$\lambda=1$, $\epsilon=7$, 2064-2077; $\lambda=4$, $\epsilon=7$, 6058-6071;
$\lambda=6$, $\epsilon=8$, 7696-7708; $\lambda=3$, $\epsilon=9$, 3733-3743;
$\lambda=5$, $\epsilon=9$, 5836-5846; $\lambda=7$, $\epsilon=9$, 7939-7950;
$\theta=2$, $\lambda=2$, $\epsilon=6$, 3911-3926; $\lambda=1$, $\epsilon=7$, 2061-2074;
$\lambda=3$, $\epsilon=9$, 3729-3739;
$\theta=3$, $\lambda=1$, $\epsilon=7$, 2058-2071;
$\theta=4$, $\lambda=1$, $\epsilon=7$, 2053-2070.

(v) Cutting out from (iv) everything disallowed by (iii), there remain the possibilities, $Q \times D$,

951 × 7950; 851 × 7701, 7703, 7705, 7707;
652 × 3926; 753 × 2067, 2069; 754 × 2064.

On trial only one of these gives × 55, etc., in dividend and × × 5 × × in line (2). The solution is

3926) 2559752 (652.

THE THREE THREES.

419) 11313 (27. 27 . 419 = 11313.

THE FOUR SEVENS (I).

959 . 713 = 683767, 959) 683767 (713.
253 . 739 = 186967,
253 . 729 = 184437,
253 . 719 = 181907.

THE FOUR SEVENS (II).

337 . 3257 = 1097609, 3257) 1097609 (337.
638 . 1595 = 1017610.

THE SEVEN SIXES.

761) 469272 (616, Remainder = 496.

THE TWO SIXES.

Let the root be $a6cde$, and let the lines be numbered (1), (2), ... (10) in order from the top.

(i) From lines (3) and (4) and the 6 in the root,

$$6(20 . a+6) < 10^3, \quad 7(20 . a+7) \geqslant 10^3,$$

hence $a = 8$.

(ii) Similarly from lines (5) and (6),

$$c(20 . 86+c) < 10^4, \quad (c+1)(20 . 86+c+1) \geqslant 10^4,$$

whence $c = 5$.

(iii) Evidently $e = 4$ or 6, and only the latter satisfies, line (10),

$$e(20 . 865d+e) > 10^7.$$

(iv) Finally $d = 8$ or 9, for anything else makes line (5) less than 10000. The two solutions are

$$\sqrt{7497135396} = 86586,$$
$$\sqrt{7498867216} = 86596.$$

THE TWO FOURS.

$$\sqrt{3445125204} = 44354 \text{ (vi)}.$$

The University, Leeds. W. E. H. BERWICK.

---

96. In Sir Andrew Noble's first paper contributed to the Royal Artillery Institution in 1858, entitled *On the Application of the Theory of Probabilities to Artillery Practice*, he used Encke's fundamental formula giving the functional form of a "probable error." Much experimental work was involved. . . . He concludes with the remark: "There is perhaps no branch of mathematics from which more information to practical artillerymen can be obtained than from the Theory of Probabilities."

97. Cornelia, wife of Pompey, whom she survived: "She had many charms apart from youthful beauty; she was well versed in literature, in playing the lyre, and in geometry, and had been accustomed to listen to philosophical discourses with profit. In addition, she had a nature which was free from that unpleasant officiousness which such accomplishments are apt to impart to young women."—*Plutarch*.

98. If the brain of the most stupid peasant were to become like that of Gauss, the peasant would develop into a mathematician like Henri Mondeux or young Colbarn.—*Taine* (in letter to Max Müller). [Calculating prodigies. Zerah Colburn.]

PARTIAL FRACTIONS ASSOCIATED WITH QUADRATIC FACTORS.

1. THE determination of the group of partial fractions corresponding to the powers of a linear factor $x+b$ in the denominator of a rational function $f(x)/\{(x+b)^n\varphi(x)\}$ is one of the comparatively rare algebraic problems whose theoretical solutions are feasible in practice. The numerator of $(x+b)^{n-r}$ is the coefficient of y^r in the expansion of $f(-b+y)/\varphi(-b+y)$, and the calculation of the group of numerators is effected by means of two Horner transformations followed by one division, which can of course be arranged synthetically; no operation need be taken further than the term in y^{n-1}, whatever the degrees of $f(x)$ and $\varphi(x)$.

The case of an unresolved quadratic factor is different, and a comparison of various processes from the standpoint of practicability is not without interest. If the function to be decomposed is $f(x)/\{X^n\varphi(x)\}$, where X denotes a quadratic function and $f(x)$, $\varphi(x)$ are polynomials prime to X, the problem is the calculation of $2n$ coefficients $A_0, B_0, A_1, B_1, \ldots$ such that

$$(A)\qquad \frac{f(x)}{X^n\varphi(x)}\equiv\Sigma\frac{A_rx+B_r}{X^{n-r}}+\frac{g(x)}{\varphi(x)};$$

the summation is for values of r from 0 to $n-1$, and $g(x)$ is a polynomial which we may or may not desire to know.

2. We remark first that the problem is not solved by the use of the irrational or complex roots of X. The substitution of $-b+y$ for x in a polynomial involves about three times as many entries for a complex value of b as for a real value. But apart from this objection, which is not conclusive, the sum of two complementary terms $C_r/(x+b)^{n-r}$, $C_r'/(x+b')^{n-r}$ is *not* usually of the form $(A_rx+B_r)/X^{n-r}$, and all that is actually accomplished by the use of the roots of X is the segregation of the *sum* $\Sigma(A_rx+B_r)/X^{n-r}$, that is, the expression of the original function by the identity

$$(B)\qquad \frac{f(x)}{X^n\varphi(x)}\equiv\frac{F(x)}{X^n}+\frac{g(x)}{\varphi(x)},$$

where $F(x)$ is a polynomial of degree $2n-1$.

It is to be admitted at once that if the use of the roots of X was an economical device for finding $F(x)$ we could be satisfied to begin in this way; the resolution of $F(x)/X^n$ is comparatively simple. Unfortunately, to find $F(x)$ as

$$\Sigma X^r\{C_r(x+b')^{n-r}+C_r'(x+b)^{n-r}\}$$

is wofully extravagant if no use is to be made of this peculiar form of the polynomial.

In passing let us refer to another direct method of finding $F(x)$. By expressing $\varphi(x)/X^n$ as a simple continued fraction whose elements are polynomials and calculating the penultimate convergent, we can determine two polynomials $\psi(x)$, $G(x)$ such that identically

$$\varphi(x)G(x)-\psi(x)X^n=1.$$

Then
$$\frac{f(x)}{X^n\varphi(x)}=\frac{f(x)G(x)}{X^n}-\frac{f(x)\psi(x)}{\varphi(x)},$$

and therefore $F(x)$ is the remainder when $f(x)G(x)$ is divided by X^n. Theoretically this process leaves nothing to be desired; it provides indeed the most elementary proof of the existence of partial fractions. Anyone who supposes that the method is tolerable in numerical examples is recommended to calculate in this way the function $F(x)$ when X^n, $\varphi(x)$ are $(1-1+3)^2$, $(2+3)^3$ and $f(x)$ is $6+9-82-316-209-141+63$.

3. In its crudest form, the method that treats $A_0, B_0, A_1, B_1, \ldots$ as undetermined coefficients in the identity

$$(C)\qquad \varphi(x)\Sigma(A_rx+B_r)X^r+X^ng(x)\equiv f(x)$$

is at a double disadvantage. Simple in theory, the solution of simultaneous linear equations with numerical coefficients, whether by means of determinants or by a process of successive elimination, is in practice tedious and hazardous if more than three or four variables are involved. Also, the coefficients of $g(x)$ *must* be introduced explicitly, although of course if they are not wanted the solution may begin with their elimination.

Perhaps for these reasons, this use of undetermined coefficients is advocated by nobody except as a last resort, and for practical purposes is entirely superseded by methods of dealing with the identity (C) which avoid the two difficulties. When $X=0$, this identity implies

$$\varphi(x)\,(A_0x+B_0)=f(x);$$

but if $X\equiv x^2+px+q$, the substitution $x^2=-px-q$, which is equivalent to $X=0$, can be used to reduce $x\varphi(x)$ and $\varphi(x)$ to linear functions $\alpha x+\beta$, $\gamma x+\delta$, and $f(x)$ to a linear function $\lambda_0x+\mu_0$. The condition

$$(\alpha x+\beta)\,A_0+(\gamma x+\delta)\,B_0=\lambda_0x+\mu_0$$

can be satisfied for the *two* roots of X only if simultaneously

$$\alpha A_0+\gamma B_0=\lambda_0,\quad \beta A_0+\delta B_0=\mu_0,$$

and this pair of equations determines A_0 and B_0. Again, because

$$f(x)-\varphi(x)\,(A_0x+B_0)$$

is zero for both roots of X, the function X is a factor of this difference, and when A_0 and B_0 are known, direct division gives a polynomial $f_1(x)$ such that *identically*

$$f(x)-\varphi(x)\,(A_0x+B_0)\equiv Xf_1(x).$$

Substituting in (C) and dividing by X, we have

$$\varphi(x)\,\Sigma(A_rx+B_r)\,X^{r-1}+X^{n-1}g(x)\equiv f_1(x),$$

where now the lowest value of r is 1, and when $X=0$,

$$\varphi(x)\,(A_1x+B_1)=f_1(x);$$

thus $$\alpha A_1+\gamma B_1=\lambda_1',\quad \beta A_1+\delta B_1=\mu_1',$$

where α, β, γ, δ have the same values as before and $\lambda_1'x+\mu_1'$ is the expression to which the repeated substitution of $-px-q$ for x^2 reduces $f_1(x)$. The operations may be continued, and the pairs of coefficients are found in succession by a uniform process which ignores the polynomial $g(x)$.

In simple examples this method is demonstrably effective. Complex numbers may be logically involved, but all the work is performed with real numbers, and an illustration of the value to the computer of supposing complex numbers to exist may be welcomed without reserve. The questions that remain open are of a severely practical kind. (1) Can the work be arranged economically? (2) If the numerical coefficients in the original function are integers, is the entry of fractional coefficients postponed to a late stage in the calculations? At first glance a negative answer to the second question seems inherent in the method: there is no reason why A_0 and B_0 should not be fractional, and in every step subsequent to the determination of these two coefficients fractions are to be expected. But it is easy to avoid this conclusion. On the hypothesis, α, β, γ, δ, λ_0, μ_0 are integers, and if η denotes $|\alpha\delta-\beta\gamma|$, then A_0, B_0, and the coefficients in $f_1(x)$ are fractions whose denominators are factors of η, A_1, B_1, and the coefficients in $f_2(x)$ are fractions whose denominators are factors of η^2, and so on. Hence, if we replace A_r, B_r by A_r'/η^{r+1}, B_r'/η^{r+1}, that is, if we deal not with (C) but with the identity

(C') $$\varphi(x)\,\Sigma\eta^{n-r-1}(A_r'x+B_r')\,X^r+\eta^nX^ng(x)\equiv\eta^nf(x),$$

fractions cannot appear.

When we turn to the question of arranging the calculations, we are led at once to present the whole argument differently. The simplest systematic process of reducing a given function to a linear form by a repeated substitution of $-px-q$ for x^2 is nothing but the synthetic division of the function by

x^2+px+q, that is, by X, and by a succession of such divisions we have the function expressed in the form $L_0+L_1X+L_2X^2+\dots$, where $L_0, L_1, L_2, \dots$ are all linear in x. Two direct operations of this kind give the coefficients in the developments

$$(D)\qquad \begin{cases} \varphi(x)=(\gamma_0x+\delta_0)+(\gamma_1x+\delta_1)X+(\gamma_2x+\delta_2)X^2+\dots, \\ f(x)=(\lambda_0x+\mu_0)+(\lambda_1x+\mu_1)X+(\lambda_2x+\mu_2)X^2+\dots, \end{cases}$$

and from the first of these we have

$$x\varphi(x)=(\alpha_0x+\beta_0)+(\alpha_1x+\beta_1)X+(\alpha_2x+\beta_2)X^2+\dots,$$

where

$$\alpha_0=\delta_0-p\gamma_0,\quad \alpha_1=\delta_1-p\gamma_1,\quad \alpha_2=\delta_2-p\gamma_2,\dots,$$
$$\beta_0=\quad -q\gamma_0,\quad \beta_1=\gamma_0-q\gamma_1,\quad \beta_2=\gamma_1-q\gamma_2,\dots;$$

if these expressions are substituted in (C), each power of X from X^0 to X^{n-1} is multiplied on each side only by a linear function of x, and equating the various linear functions we have

$$(\alpha_0x+\beta_0)A_0+(\gamma_0x+\delta_0)B_0 \qquad\qquad =\lambda_0x+\mu_0,$$
$$(\alpha_0x+\beta_0)A_1+(\gamma_0x+\delta_0)B_1+(\alpha_1x+\beta_1)A_0+(\gamma_1x+\delta_1)B_0 =\lambda_1x+\mu_1,$$
$$(\alpha_0x+\beta_0)A_2+(\gamma_0x+\delta_0)B_2+\sum_{1,2}\{(\alpha_sx+\beta_s)A_{2-s}+(\gamma_sx+\delta_s)B_{2-s}\}=\lambda_2x+\mu_2,$$
$$(\alpha_0x+\beta_0)A_3+(\gamma_0x+\delta_0)B_3+\sum_{1,2,3}\{(\alpha_sx+\beta_s)A_{3-s}+(\gamma_sx+\delta_s)B_{3-s}\}=\lambda_3x+\mu_3,$$

and so on. That is, A_r, B_r are given by

$$(E)\qquad \alpha A_r+\gamma B_r=\lambda_r',\quad \beta A_r+\delta B_r=\mu_r',$$

where α, β, γ, δ, λ_0', μ_0' coincide with α_0, β_0, γ_0, δ_0, λ_0, μ_0, and for values of r from 1 to $n-1$,

$$(F)\qquad \lambda_r'=\lambda_r-\Sigma(\alpha_sA_{r-s}+\gamma_sB_{r-s}),\quad \mu_r'=\mu_r-\Sigma(\beta_sA_{r-s}+\delta_sB_{r-s}),$$

the summations being for the values of s from 1 to r.

4. There is a method essentially more direct than that just described. For this we suppose the factor X to have the form $x^2+2bx+c$. Then substitution of a new variable y for $x+b$ gives the function the form

$$f(-b+y)/\{(y^2+k)^n\varphi(-b+y)\},$$

where k has the value $c-b^2$.

Were it not for direct evidence to the contrary in worked examples published over well-known names, we might have thought it obvious that the problem of separating from a function of the particular form

$$F(y^2)/\{(y^2+k)^n\Phi(y^2)\}$$

the partial fractions of the form $K_r/(y^2+k)^{n-r}$ is the same as that of extracting from $F(x)/\{(x+k)^n\Phi(x)\}$ the elements associated with powers of $x+k$: the numerator K_r is the coefficient of z^r in the expansion of $F(-k+z)/\Phi(-k+z)$.

But whatever the polynomials $f(x)$, $\varphi(x)$, the product $\varphi(-b+y)\varphi(-b-y)$ is of the form $\Phi(y^2)$, and the product $f(-b+y)\varphi(-b-y)$ is expressible by separation of the odd powers of y from the even in the form $F'(y^2)+yF''(y^2)$. That is to say, the original function needs only two Horner transformations and two multiplications to throw it into the form

$$\frac{F'(y^2)}{(y^2+k)^n\Phi(y^2)}+y\frac{F''(y^2)}{(y^2+k)^n\Phi(y^2)}.$$

Three more Horner transformations and two divisions give the coefficients in the expansions of

$$F'(-k+z)/\Phi(-k+z)\quad\text{and}\quad F''(-k+z)/\Phi(-k+z),$$

and if these are K_0', K_1', ... and K_0'', K_1'', ..., a typical fraction in the decomposition of

$$f(-b+y)/\{(y^2+k)^n\Phi(-b+y)\}\quad\text{is}\quad (K_r'+yK_r'')/(y^2+k)^{n-r},$$

and a typical element of the original function is

$$\{K_r''x+(K_r'+bK_r'')\}/X^{n-r}.$$

In comparison with the method of § 3, this process suffers at two points. If the quadratic factor is given in the first place as ux^2+vx+w, where u, v, w are integers, the substitution $\xi=ux$ is sufficient to replace the factor by one of the form $\xi^2+p\xi+q$, but it may be necessary to take $\xi=2ux$ if the form required is $\xi^2+2b\xi+c$. Thus the later method is liable to involve powers of 2 avoided in the earlier. Also, the three functions $F'(y^2)$, $F''(y^2)$, $\Phi(y^2)$ must be determined completely, whatever the value of n; since the number of entries in a real Horner transformation applied to a polynomial of degree p approximates to p^2 if the transformation is complete and to $n(2p-n)$ if only the terms of degree not greater than n are wanted, it is a serious drawback to be compelled to deal at full length with $f(x)$ and $\varphi(x)$ even when n is small.

On the other hand, the method of § 4 is the more straightforward. The operations are restricted in number and of a type so familiar that we have nothing to learn as to the manner of performing them economically; it is easier to arrange intelligibly the figures of a few large operations, each of which is complete in itself, than those relating to a number of small steps, such as are involved in passing from the solution of one pair of equations of the form (E) to the calculation of the coefficients for the succeeding pair. Lastly, fractional coefficients cannot appear till the work is all but finished, and there is no temptation to forestall them by introducing multipliers that may be very large and that may in fact not be required.

5. It is possible to combine the advantages of the two methods. First, we develop $f(x)$ and $\varphi(x)$ as in (D) as far as the terms in X^{n-1}; in this work we keep X in the form x^2+px+q, it being immaterial whether p is odd or even. Next, if p is odd, we make the substitution $\xi=2x$ to replace X by

$$\tfrac{1}{4}(\xi^2+2p\xi+4q);$$

the change is trivial, and to avoid accumulating symbols we will describe the remainder of the process as if this step was unnecessary, taking X as $x^2+2bx+c$, that is, as $(x+b)^2+k$, where k is $c-b^2$.

Substitution of $-b+y$ for x in the first n terms on the right of (D) gives functions $\varphi_1(X)+y\varphi_2(X)$, $f_1(X)+yf_2(X)$, where

$$\varphi_1(X)=\Sigma(\delta_r-b\gamma_r)X^r, \quad \varphi_2(X)=\Sigma\gamma_r X^r,$$
$$f_1(X)=\Sigma(\mu_r-b\lambda_r)X^r, \quad f_2(X)=\Sigma\lambda_r X^r,$$

each summation being for values of r from 0 to $n-1$. Multiplying by $\varphi_1(X)-y\varphi_2(X)$ and replacing y^2 by $X-k$, we have the function

$$\frac{\{f_1(X)\,\varphi_1(X)+(k-X)f_2(X)\,\varphi_2(X)\}+(x+b)\{f_2(X)\,\varphi_1(X)-f_1(X)\,\varphi_2(X)\}}{X^n\,[\{\varphi_1(X)\}^2+(k-X)\{\varphi_2(X)\}^2]},$$

which has the same elementary functions associated with powers of X as the original function; to evaluate this function completely is unnecessary, for powers of X of degree higher than $n-1$ can be omitted in every multiplication. Two synthetic divisions give the coefficients denoted in § 4 by K_r' and K_r'', and we have as before

$$A_r=K_r'', \quad B_r=K_r'+bK_r''.$$

6. In conclusion we ought to recognise that while the method of § 5 ought to involve the minimum of labour, no estimate of numerical complexity can safely be applied to what may be called manufactured examples. If the function to be resolved has been composed by the addition of simple parts, so that the numbers A_r, B_r are small integers, the equations (E) and the formulae (F) *must* be abnormally simple. Even the equations that arise directly from (C) may be rendered quite manageable by some random *assumption* as to the magnitude of the coefficients to be determined; for example, the function suggested at the end of § 2 can be resolved quickly in this way if it is taken for granted that the coefficients are all integers numerically less than twenty!

E. H. Neville.

MATHEMATICAL NOTES.

598. [A. 1.] *A Simple and Elementary Method of Multiplication.*

The following example is self-explanatory:

$$\begin{array}{rrrrrrr}
9 & 9 & 7 & 2 & 4 & & \\
 & & 8 & 3 & 4 & & \\
\hline
72 & 72 & 56 & 16 & 32 & & \\
 & 27 & 27 & 21 & 06 & 12 & \\
 & & 36 & 36 & 28 & 08 & 16 \\
\hline
83 & 1 & 6 & 9 & 8 & 1 & 6
\end{array}$$

Is this original?

168 Bath Road, Southsea, Portsmouth. W. R. MEADOWS.

599. [R. 4. c. d.; V. a. μ.] What is the orthodox way of solving the following problem?

"A framework of light rods is in the form of an isosceles triangle ABC, the middle point D of the base BC being connected by light rods to E and F, the middle points of the equal sides AB, AC. The framework is maintained in a vertical plane by supports at B, C, BC being horizontal. Equal weights are hung from A, D, E, F. Draw a diagram showing the stresses in the various rods." [Sci. Schol. Ox.:—University, Balliol, Oriel, etc. Dec. 1920.]

600. [V. 2. 10.] The same paper also contains the following question:

"A body moves under an accelerating force g and against a frictional resistance. Investigate its motion when the frictional resistance varies

(1) directly as the velocity v,

(2) as $av+bv^2$."

The second part of this question seems a very long piece of work for a paper of this sort to require.

Is it too much to ask for a definite pronouncement as to the amount of mathematical equipment expected from candidates for Science Scholarships at Oxford, Cambridge and elsewhere?

Dean Close School, Cheltenham. T. M. A. COOPER.

Mr. W. J. Dobbs replies as follows to the first of the above queries: (1) It is, I believe, an accepted principle, that, in designing a light jointed framework, the bars should be made so strong that there is no need to call upon any joint for a constraining couple. Rigidity in a joint becomes then an added source of strength. Hence, in drawing the stress diagram of a light jointed framework, all joints are reckoned as free, and bars connecting different joints are reckoned as separate bars.

(2) Again, "if a framework has n joints, it requires $2n-3$ bars to make it rigid." "In a strictly indeformable figure $s=2v-3$," s being the number of sides and v the number of vertices.

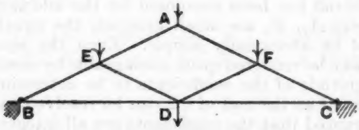


FIG. 1.

In agreement with (1) the given framework has 6 joints but only 8 bars. It is therefore, by (2), not strictly indeformable, having one bar short. But it may be contended that if the joints B and C are both fixed, there is added in effect a ninth bar BC. Yes; but $BDCB$ does not form a triangle; also the supporting forces at B and C cannot be vertical. Without appreciably increasing the lengths of BD and DC, the joint D may sink appreciably. The space and stress diagrams are then as shown below. When BD and DC each $\to$ horizontal lines, RL and TQ each $\to \infty$, *i.e.* the tension in each tie-bar $\to \infty$.

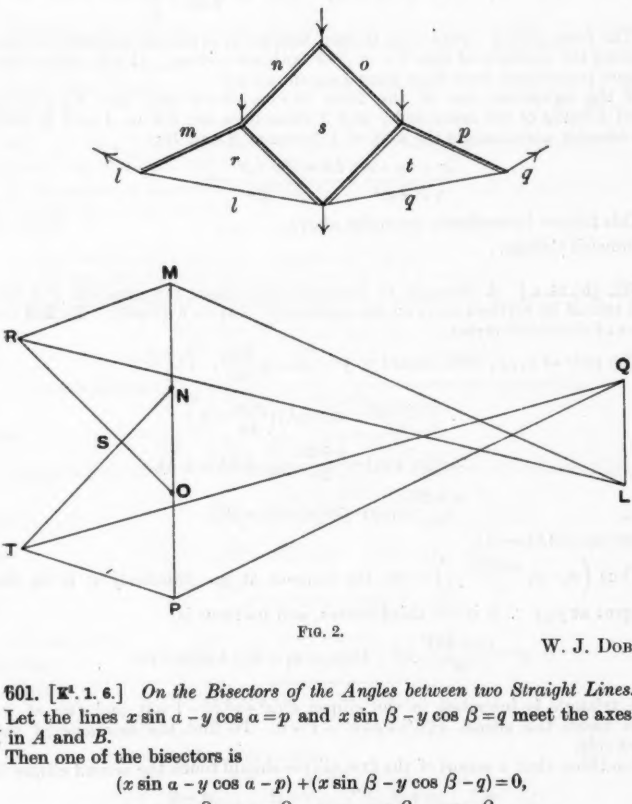


FIG. 2.

W. J. DOBBS.

601. [K[1]. 1. 6.] ***On the Bisectors of the Angles between two Straight Lines.***

Let the lines $x \sin \alpha - y \cos \alpha = p$ and $x \sin \beta - y \cos \beta = q$ meet the axes of X in A and B.

Then one of the bisectors is

$$(x \sin \alpha - y \cos \alpha - p) + (x \sin \beta - y \cos \beta - q) = 0,$$

$$\textit{i.e.}\ 2x \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} - 2y \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} = p + q,$$

$$\textit{i.e.}\ x \sin \frac{\alpha+\beta}{2} - y \cos \frac{\alpha+\beta}{2} = \frac{p+q}{2 \cos \frac{\alpha-\beta}{2}}, \quad \text{..................(I.)}$$

an since this makes an angle $\frac{\alpha+\beta}{2}$ with OX, it cuts the axis between A and B.

The bisector which cuts the axis of X outside the segment AB is

$$(x \sin \alpha - y \cos \alpha - p) - (x \sin \beta - y \cos \beta - q) = 0,$$

reducing to
$$x \cos \frac{\alpha+\beta}{2} + y \sin \frac{\alpha+\beta}{2} = \frac{p-q}{2 \sin \frac{\alpha-\beta}{2}}$$

or
$$x \sin\left(\frac{\pi+\alpha+\beta}{2}\right) - y \cos\left(\frac{\pi+\alpha+\beta}{2}\right) = \frac{p-q}{2 \sin \frac{\alpha-\beta}{2}}.$$

[The form $x \sin \alpha - y \cos \alpha = p$ is used because it saves the necessity of considering the question of sign if α or β or both are obtuse. It is in other ways a more convenient form than $x \cos \alpha + y \sin \alpha = p$.]

If the equations are of the form $lx+my+n=0$ and $Lx+My+N=0$, l and L being of the same sign; and if these lines cut OX in A and B, then the bisector which meets the axis of X between A and B is

$$\frac{lx+my+n}{\sqrt{l^2+m^2}} + \frac{Lx+My+N}{\sqrt{L^2+M^2}} = 0.$$

This follows immediately from the above. F. C. B.

Dulwich College.

602. [L^1. 14. a.] A triangle is circumscribed about the parabola $y^2=4ax$, and two of its vertices move on the confocal $y^2=4(a+\lambda)(x+\lambda)$. To find the locus of the third vertex.

The pole of y_1, y_2 with regard to $y^2=4ax$ is $\frac{y_1y_2}{4a}$, $\frac{y_1+y_2}{2}$;

$$\therefore \quad \frac{(y_1+y_2)^2}{4} = 4(a+\lambda)\left(\frac{y_1y_2}{4a}+\lambda\right);$$

$$\therefore \quad a(x_1+x_2) = \frac{a+2\lambda}{2a} y_1y_2 + 4\lambda(a+\lambda),$$

or
$$\frac{a+2\lambda}{2a} y_1 \cdot y_2 = 2a[x_2+(x_1-\mu)],$$

where $a\mu = 4\lambda(a+\lambda)$.

Thus $\left(x_1-\mu, \frac{a+2\lambda}{a} y_1\right)$ is on the tangent at y_2. Similarly it is on the tangent at y_3; $\therefore$ it is the third vertex, and its locus is

$$y^2 = \frac{(a+2\lambda)^2}{a^2} y_1^2 = 4(a+\mu)x_1 = 4(a+\mu)(x+\mu).$$

A triangle is inscribed in the ellipse $x^2/a^2+y^2/b^2-1=0$, and two of its sides touch the ellipse $x^2/a'^2+y^2/b'^2-1=0$. To find the envelope of the third side.

Condition that a secant of the first ellipse should touch the second ellipse is

$$\frac{a'^2}{a^2}\cos^2\frac{\alpha+\beta}{2} + \frac{b'^2}{b^2}\sin^2\frac{\alpha+\beta}{2} = \cos^2\frac{\alpha-\beta}{2},$$

i.e. $$p(1+\cos\overline{\alpha+\beta}) + q(1-\cos\overline{\alpha+\beta}) = 1+\cos(\alpha-\beta) \quad \left(p \equiv \frac{a'^2}{a^2},\ q \equiv \frac{b'^2}{b^2}\right),$$

i.e. $$(p+q-1)+(p-q-1)\cos\alpha\cos\beta-(p-q+1)\sin\alpha\sin\gamma=0.$$

Similarly
$$(p+q-1)+(p-q-1)\cos\alpha\cos\gamma-(p-q+1)\sin\alpha\sin\gamma=0.$$

These are the conditions that the points β and γ should be on the line

$$(p+q-1)+(p-q-1)\frac{x\cos a}{a}-(p-q+1)\frac{y\sin a}{b}=0.$$

This, then, is the equation of the third side, and its envelope is

$$\frac{x^2}{a^2}(p-q-1)^2+\frac{y^2}{b^2}(p-q+1)^2=(p+q-1)^2,$$

or $$(p^2-2pq+q^2-2p-2q+1)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+4pq\left(\frac{x^2}{a'^2}+\frac{y^2}{b'^2}-1\right)=0.$$

Similarly for the reciprocal problem, and any other similar problem in which the coordinates of a point on the first conic can be expressed in terms of a variable parameter.

[*v.* Smith's *Conics*, p. 327, for another solution; cf. Nos. 11, p. 433; 33, p. 439.]

N. M. GIBBINS.

603. [L[1]. 16. a.] *To find the other sides of a quadrilateral determined* (1) *by a point, line and conic,* (2) *two lines and conic.*

(1) Equation of lines joining $O(x', y')$ to the points A, B, in which the line $L=0$ cuts the conic $S=0$, is

$$L'^2S-2LL'P+L^2S'=0. \qquad \text{(i)}$$

If these cut S again in C, D, equation of CD is $LS'-2L'P=0$.

If AB and DC meet in Q, equation of OQ is $LS'-L'P=0$.

Again, $R(x_1y_1)$, point of intersection of AC and BD, is the pole of OQ;

$$\therefore\ \frac{ax_1+h_1y+g}{S'l-L'X'}=\frac{hx_1+by_1+f}{S'm-L'Y'}=\frac{gx_1+fy_1+c}{S'n-L'Z'};$$

$$\text{each ratio}=\frac{\Delta x_1}{S'(Al+Hm+G)-L'\Delta x'}=\text{two similar expressions,}$$

also $$=\frac{\Delta L_1}{S'\Sigma-L'^2\Delta}=\frac{P_1}{LS'-L'P}=\frac{S_1}{S'L_1}\equiv K.$$

Applying equation (i) to the point (x_1y_1), the equation of AC and BD is

$$L_1^2S-2L_1LP_1+S_1L^2=0,$$

i.e. $$L_1S-2LP_1+S'KL^2,\ \text{for}\ S_1=S'L_1K,$$

or $$\left(\frac{S'\Sigma}{\Delta}-L'^2\right)S-2L(LS'-L'P)+L^2S'=0,$$

i.e. $$\frac{S'\Sigma}{\Delta}S=L'^2S-2LL'P+L^2S'.$$

(2) Let now $$LS'-2L'P\equiv l'x+m'y+n'\equiv M,$$

so that $$2L'P=LS'-M\quad\text{and}\quad L'S'+M'=0.$$

Then

$$\frac{X'}{lS'-l'}=\frac{Y'}{mS'-m'}=\frac{Z'}{nS'-n'}=\frac{\Delta x'}{S'(Al+Hm+Gn)-(Al'+Hm'+Gn')}$$

$=$two similar expressions

$$=\frac{\Delta L'}{S'\Sigma-\Pi}=\frac{\Delta M'}{S'\Pi-\Sigma'}=\frac{\Delta(M'+L'S')}{S'^2\Sigma-\Sigma'}=\frac{P}{S'L-M}=\frac{1}{2L'}.$$

Lines AD and BC are $L'^2S+LM=0$ or $(\Pi-S'\Sigma)S=2\Delta LM$,

and lines AC and BD are $L'^2S+LM=\dfrac{S'\Sigma}{\Delta}S$ or $(\Pi+S'\Sigma)S=2\Delta LM$.

Combined equation of the four lines is

$$(\Pi S-2\Delta LM)^2=\Sigma\Sigma'S^2,\ \text{for}\ S'^2=\Sigma'/\Sigma.$$

N. M. GIBBINS.

604. [J. 1.] *A Mathematical Solution of the "Daily Mail" Puzzle.*

The *Daily Mail* Puzzle consists of nineteen hexagons, numbered from 1 to 19, around which are arranged in various ways the six colours red, orange, blue, yellow, green, and violet. The problem is to fit the hexagons together, matching the colours to produce the arrangement:

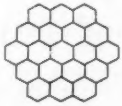


FIG. 1.

Make a list of the nineteen hexagons, denoting the colours

	red,	orange,	blue,	yellow,	green,	violet	
by the letters	a,	b,	c,	d,	e,	f	respectively,

starting with red and working in a clockwise direction.

1., 12. $a\ b\ c\ d\ e\ f$	6. $a\ d\ e\ f\ b\ c$	14. $a\ c\ e\ d\ f\ b$
2. $a\ f\ b\ d\ e\ c$	7. $a\ f\ d\ e\ c\ b$	15. $a\ b\ f\ d\ e\ c$
3. $a\ e\ b\ c\ d\ f$	8. $a\ b\ d\ e\ f\ c$	16. $a\ e\ f\ d\ c\ b$
4., 10. $a\ c\ e\ b\ f\ d$	9. $a\ c\ e\ d\ b\ f$	17. $a\ b\ c\ e\ f\ d$
	11. $a\ e\ d\ c\ f\ b$	18. $a\ c\ d\ b\ e\ f$
5. $a\ b\ c\ f\ e\ d$	13. $a\ e\ f\ c\ b\ d$	19. $a\ f\ e\ b\ c\ d$

There are thus seventeen different arrangements. The hexagons having in common the combinations *ab*, *ac*, *ad*, etc., can now be written down.

ab	1, 12, 5, 8, 15, 17.	*ca*	2, 6, 8, 15.	*ea*	——
ac	4, 10, 9, 14.	*cb*	7, 13, 16.	*eb*	3, 4, 10, 19.
ad	6.	*cd*	1, 12, 3, 18, 19.	*ec*	2, 7, 15.
ae	3, 11, 13, 16.	*ce*	4, 10, 9, 14, 17.	*ed*	5, 9, 11, 14.
af	2, 7, 19.	*cf*	5, 11.	*ef*	1, 12, 6, 8, 13, 16, 17, 18.
ba	7, 11, 14, 16.	*da*	4, 10, 5, 13, 17, 19.	*fa*	1, 12, 3, 9, 18.
bc	1, 12, 3, 5, 6, 17, 19.	*db*	9, 18.	*fb*	2, 6, 11, 14.
bd	2, 3, 13.	*dc*	11, 16.	*fc*	8, 13.
be	18.	*de*	1, 12, 2, 6, 7, 8, 15.	*fd*	4, 10, 7, 15, 16, 17.
bf	4, 10, 9, 15.	*df*	3, 14.	*fe*	5, 19.

Consider the centre hexagon. Let AB be e. AD cannot be a, because the combination *ea* does not exist. If AC is a, AD may be b, c, d or f (4 cases).

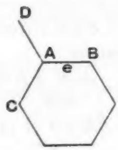


FIG. 2.

If AC is not a, AD cannot be e, a, or the letter at AC, so that it is one of the three remaining letters (3 cases). The combination *ae* occurs four times (3, 11, 13, 16), so that the problem resolves itself into $4 \times 4 + 13 \times 3 = 55$ cases.

The centre hexagons giving rise to four cases (3, 11, 13, 16) are those most likely to lead to a solution. It will be found that there is only one solution—16 as centre hexagon and AD as c. The manner in which the hexagons are fitted together to solve the problem is explained by the table and diagram given below. The numbers in brackets in column (3) denote available hexagons tried and discarded. The fact that there is only one solution was

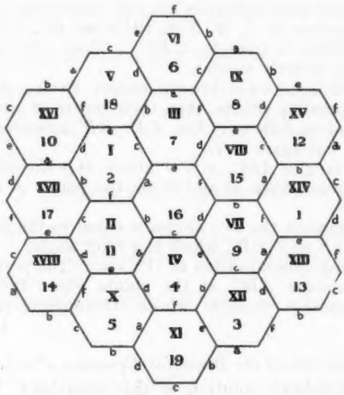


FIG. 3.

confirmed—all the solutions received by the *Daily Mail* differed only by the transposition of the duplicate hexagons 1 and 12 and 4 and 10.

(1) Sequence of placing hexagons.	(2) Combination required.	(3) Hexagons available.
I.	*ca*	2, (6), (8), (15)
II.	*fb*	(6), 11, (14)
III.	*ec*	7, (15)
IV.	*ac*	4 [or 10], (9), (14)
V.	*be*	18
VI.	*ad*	6
VII.	*ed*	(5), 9, (14)
VIII.	*fd*	(10), 15, (17)
IX.	*fc*	8, (13)
X.	*ed*	5, (14)
XI.	*af*	19
XII.	*bc*	3
XIII.	*da*	(10), 13, (17)
XIV.	*fa*	1 [or 12]
XV.	*cd*	12
XVI.	*fd*	10, (17)
XVII.	*bc*	17
XVIII.	*df*	14

T. L. LIZINS.

605. [K^1. 2. a.] *A Generalisation of a Theorem in Geometry.*

We have the theorem in Elementary Geometry, viz.: If the feet of the perpendiculars from a point P to the sides of a triangle ABC are collinear, then P lies on the circumcircle of ABC. ....(1)

In other words, let a transversal cut the sides of a triangle ABC at L, M, N. Then, if the perpendiculars at L, M, N, to the sides on which they lie are concurrent, the point of concurrence must lie on the circumcircle of ABC. What happens if these perpendiculars are not concurrent?

Let the perpendiculars at L, M, N to the sides BC, CA, AB respectively, on which they lie, form a triangle $A'B'C'$. Then ABC and $A'B'C'$ are in perspective and also directly similar.

Now, if two plane figures are directly similar, having A, B and A', B' for two pairs of corresponding points, then their centre of similitude (or double point) D is obtained as follows: Let AA', BB' intersect at O. Then the circles AOB, $A'OB'$ cut again at D.

Thus, if for the triangles ABC, $A'B'C'$ above, O is the centre of perspective and D the double point, then D and O are the points of intersection of the circles ABC, $A'B'C'$.

Also, the angle between the two circles is equal to the angle ADA', *i.e.* to the angle between AB and $A'B'$, which is a right angle.

Hence the following generalisation of (1), viz.: The perpendiculars drawn to the sides of a triangle ABC, at the points where they are cut by any transversal, form another triangle whose circumcircle cuts that of ABC orthogonally.

G. A. SRINIVASAN.

606. [A. 3.] *A Solution of the Binomial Equation $x^{17}=1$.*

May I offer an algebraic solution of this equation? I have found this form useful in dealing with the geometrical aspect of this problem:

(I) From $x^{17}=1$, we have directly that $x^{16}=\frac{1}{x}$, $\frac{1}{x^{16}}=x$, and that

$$x^{16}+\frac{1}{x^{16}}=x+\frac{1}{x};$$

consequently, putting

$$\left(x+\frac{1}{x}\right)=a,\quad \left(x^2+\frac{1}{x^2}\right)=b,\quad \left(x^4+\frac{1}{x^4}\right)=c,\quad \left(x^8+\frac{1}{x^8}\right)=d,$$

we obtain the relations

$$a^2=2+b,\quad b^2=2+c,\quad c^2=2+d,\quad d^2=2+a.$$

(II) Subtracting, we get

$$a^2-c^2=b-d,\quad b^2-d^2=c-a,$$

from which, by multiplication, we get

$$(a+c)\times(b+d)=-1;$$

and putting $(a+c)=e$, then $(b+d)=-\frac{1}{e}$.

(III) Again, adding, we get

$$a^2+c^2=4+b+d,\quad b^2+d^2=4+a+c,$$

or

$$(a-c)^2=8-\frac{2}{e}-e^2,\quad (b-d)^2=8+2e-\frac{1}{e^2}.$$

(IV) Hence we can write either of the first two equations in (II) in terms of e, as

$$\pm e\sqrt{8-\frac{2}{e}-e^2}=\sqrt{8+2e-\frac{1}{e^2}}.$$

And rationalising, and dividing by e, we obtain the symmetrical equation

$$e^3+0-8e+4+\frac{8}{e}+0-\frac{1}{e^3}=0.$$

Now, by substituting $f=e-\frac{1}{e}$, we get

$$f^3+0-5f+4=0,$$

or

$$(f-1)(f^2+f-4)=0.$$

(V) Equating the second factor to 0, we obtain $f=\frac{-1\pm\sqrt{17}}{2}$, from which numerical values of $e, a, b, \ldots$, and eventually of x itself, may be obtained.

[*Note.*—The factor $(f-1)=0$ is evidently connected with the equation $x^{15}=1$, this being involved in our solution, as that $x^{16}+x^{-16}=x+x^{-1}$ is equally true in that case.] C. H. CHEPMELL.

607. [**K**[1]. **21. b.**] *A Method of Trisecting any Angle.*

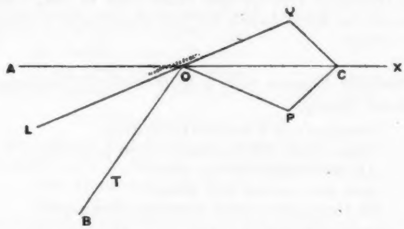
OX is a straight rod.

LOQ, OP and OT are rods which can turn about a vertical axis at O.

The rod LOQ carries a small vertical plane mirror at O.

OP carries a vertical pin.

OT carries two vertical pins along its axis.



CQ and CP are equal straight rods pivoted at C, and C can slide along OX.

CQ and CP are pivoted to OQ and OP at Q and P respectively.

Thus, as C moves along OX, the rods LOQ and OP turn about O, so that

$$C\hat{O}Q=C\hat{O}P.$$

Suppose AOB is the angle which it is required to trisect.

OX is fixed so that AOX is a straight line.

OT is fixed along the other arm, OB, of the angle AOB.

C is moved along OX until the two pins in OT are in line with the image of the pin in OP.

Thus PO and OT are incident and reflected rays in the plane mirror.

Hence $$P\hat{O}Q=T\hat{O}L=B\hat{O}L.$$

But $$P\hat{O}C=Q\hat{O}C=A\hat{O}L;$$

$$\therefore\ P\hat{O}Q=2A\hat{O}L,$$

$$i.e.\ B\hat{O}L=2A\hat{O}L.$$

Hence the angle AOB is trisected.

To ensure greater accuracy, the pin on OP could be replaced by cross wires, and the two pins on OT by a small telescope. A. W. LUCY.

608. [V.] *The Pillory*, ii. (pp. 160, 204 of this volume).

Prof. Mathews' assertion that this problem is a good example of cases in which pure geometry is a more effective instrument than analytical is beyond dispute, but by his choice of axes he fails to give the analytical devil his due.

Let us use oblique axes BC, BA and identify points by their projections, not by their coordinates. Writing ω for the angle CBA, we have the projections of C as $(a, a\cos\omega)$, of A as $(c\cos\omega, c)$, and if we take those of O as (p, q), one form of the relation between p and q is

$$(p-a)(q-c)=(p-c\cos\omega)(q-a\cos\omega), \qquad \text{(i)}$$

which expresses that OA and OC are parallel. Also (u, q) will serve for *any* point U on OK, and (p, v) for *any* point V on OH. The condition for B, U, V to be collinear is

$$uv=pq, \qquad \text{(ii)}$$

and the condition for AU, BV to be parallel is

$$(u-c\cos\omega)(v-a\cos\omega)=(p-a)(q-c),$$

which (i) enables us to replace by

$$(u-c\cos\omega)(v-a\cos\omega)=(p-c\cos\omega)(q-a\cos\omega). \qquad \text{(iii)}$$

It is obvious: (1) that the conditions (ii), (iii) cannot be identical unless the coefficients of u and v in the latter condition vanish, that is, unless $\cos\omega$ is zero; (2) that if $\cos\omega$ is zero, (iii) does reduce to (ii); (3) that for a value of $\cos\omega$ other than zero the conditions (ii), (iii) are satisfied simultaneously by $u=p$, $v=q$, and therefore also by one other pair of real values of u and v, which *inspection* of the form taken by (iii) when pq/u is written for v shows to be $u=cq/a$, $v=ap/c$. E. H. Neville.

609. [R.] *Relativity Rhymes, with a Mathematical Commentary.*

A. *The Restricted Theory.*

Einstein's is a wonderful notion
That a rod will contract when in motion,
All the clocks will go slow,
And yet no one will know!
So the matter need cause no commotion.

B. *The General Theory, applied to Planetary Motion.*

If the path of a planet you'd trace,
You've Christoffel's weird symbols to face,
For an orbit, you see
Is as straight as can be
On a surface in quintuple space.

Mathematical Commentary.

Lines 2 and 3. This is as stated in the English translation of Einstein's popular book. However, Eddington distinctly states that these effects are only apparent.

Line 4. *This is the essence of the restricted theory.* Stated more fully it is the *assumption* that: It is impossible by any conceivable experiment for an observer to detect his uniform motion with respect to the "ether."

Line 5. Because all experiments (with the exception of that with Fizeau's Water Tube) yield merely negative results.

Line 7. The restricted theory leads to the conclusion that although two observers, moving relatively to each other with any uniform velocity, will disagree about the measurement of space and time, they will agree in their estimate of the velocity of light c and also of the so-called "interval between two events," $\sqrt{(c^2dt^2-dx^2-dy^2-dz^2)}$, where t is time and x, y, z cartesian coordinates. The general theory drops the assumed agreement about the velocity of light, but *assumes* that agreement still exists about an interval ds:

The four coordinates $x_1 x_2 x_3 x_4$ are *collectively* a representation of space and time, but they may be so in any of an infinite number of ways.

ds^2 is of the second degree in dx_1, dx_2, dx_3, dx_4. The complete expression contains ten terms. The coefficients of these terms are *assumed* to satisfy ten relations connecting their differential coefficients with respect to the x's. These relations are spoken of collectively as " the vanishing of the contracted Riemann-Christoffel Tensor."

This is the essence of Einstein's General Theory. It is not quite as artificial as it looks, for it is the simplest set of relations that preserve the same form when subjected to a change of coordinates.

Lines 9 and 10. That is, ds satisfies the condition involving the *four* coordinates analogous to the condition involving *two* coordinates which imply that it is an element of an arc of a geodesic on a surface in three dimensions. A geodesic on a plane is a straight line, while that on a sphere is a great circle. On any (convex) surface a string stretched between two points will lie in a geodesic. The application of the Calculus of Variations will give two differential equations for the ordinary geodesic in three dimensional space, or four for our " quintuple space." These differ slightly from the equations of motion under the inverse square law, and account for the observed anomaly in the motion of the perihelion of Mercury.

H. PIAGGIO.

The following, signed " Even Chillier," have been received :

(1) *The German Jeweller's Complaint.*

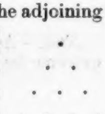
Einschtein hash a shcandaloush notion
About how the univershe gosh on.
He shaysh I sha'n't know
If my glocksh all go shlow
And my money'sh shrunk up by itsh motion.

(2) *The Reveller's Charter.*

They tell me that Einstein's been thinking
A stick can't be twirled without shrinking,
And we never could know
If our clocks *were* all slow,
So I needn't say what we've been drinking.

(3) Christoffel will help you to trace
The path of the earth in a brace
Of shakes, for you see
It's only a bee-
Line followed in quintuple space.

610. [I. 25.. b] *A Mathematical Recreation.*



If points are arranged at equal distances from each other as in the adjoining figure the number of these points is $\frac{1}{2}m(m+1)$, m being the number of the points in the base line. It is well known that the ancient Greeks used the term triangular number (ἀριθμὸς τρίγωνος) for such a number, and that the Roman surveyors sometimes used the formula $\frac{1}{2}m(m+1)$ to find the area of an equilateral triangle whose side is m units long.*

If the base line of the triangle determined by the given figure remains fixed while m is increased without limit the formula $\frac{1}{2}m(m+1)$ will continue to express the number of these points in this equilateral triangle. One might therefore be led to assume that the area of this triangle could be represented to any desired degree of approximation by means of the given

*** On page 345 of Bubnow's *Gerberti Opera Mathematica*, 1899, this formula is expressed in the following words : "Omnis trigonus aequilaterus unum latus in se multiplicat ipsum latus ad eam multiplicationem addit, horum dimidiam sumit et sic aream suam implet."**

formula, provided the unit of measure were made sufficiently small. This is especially true if one thinks of a surface as composed of points distributed symmetrically.

The fact that the formula $\frac{1}{2}m(m+1)$ does not express the area of an equilateral triangle whose sides are equal to m, even when m increases without limit, but has for its lower limit the area of an isosceles triangle whose altitude and base are both equal to m, seems therefore to be discordant with the assumption that a surface is composed of points. When this formula is used to find the area of an equilateral triangle the error will always exceed 15 per cent., as can easily be verified.

G. A. Miller.

611. [I. 1.] *A Notation for the Use of Common Logarithms.*

The definition of logs to base 10 is implied in

$$2 = 10^{\cdot 3010\ldots}.$$

This may be written $2 = {}_{10}[\cdot 3010\ldots.]$

This form is convenient in computation for two reasons:

(i) It is a continual reminder of what a logarithm is.

(ii) It enables a straight-on direct statement of the working to be made.

Example 1.

$$\begin{aligned} x &= \sqrt{\{23\cdot 14^2 + 51\cdot 06^2\}} \\ &= \sqrt{\{{}_{10}[2(1\cdot 3643)] + {}_{10}[2(1\cdot 7081)]\}} \\ &= \sqrt{\{{}_{10}[2\cdot 7286] + {}_{10}[3\cdot 4162]\}} \\ &= \sqrt{\{535\cdot 3 + 2607\}} = \sqrt{3142} = {}_{10}[\tfrac{1}{2}(3\cdot 4972)] = {}_{10}[1\cdot 7486] = \underline{56\cdot 06}. \end{aligned}$$

Example 2.

$$\begin{aligned} x &= 1\cdot 372^{3\cdot 14} \\ &= {}_{10}[3\cdot 14 \times 0\cdot 1374] \\ &= {}_{10}[{}_{10}[0\cdot 4969 + \bar{1}\cdot 1380]] = {}_{10}[{}_{10}[\bar{1}\cdot 6349]] = {}_{10}[0\cdot 4314] = \underline{2\cdot 702}. \end{aligned}$$

In the cases for which logs are commonly used, the working can be set out thus:

$$\begin{aligned} &\frac{217\cdot 6 \times \sqrt{81\cdot 93} \times 0\cdot 367^2}{9\cdot 321^3 \times \sqrt{0\cdot 5371} \times 1\cdot 629^{2\frac{1}{2}}} \\ &= {}_{10}[2\cdot 3377 + \tfrac{1}{2}(1\cdot 9134) + 2(\bar{1}\cdot 5647)] \\ &\qquad - 3(0\cdot 9695) - \tfrac{1}{2}(\bar{1}\cdot 7537) - 2\tfrac{1}{2}(0\cdot 2119)] \\ &= {}_{10}[2\cdot 3377 - 2\cdot 9085 \\ &\qquad 0\cdot 9567 \quad \bar{1}\cdot 8769 \\ &\qquad \bar{1}\cdot 1294 \quad 0\cdot 5298] \\ &= {}_{10}[2\cdot 4238 - 3\cdot 3149] = {}_{10}[\bar{1}\cdot 1089] = \underline{0\cdot 1285}. \end{aligned}$$

Note.—As a point of incidental interest in logarithms, the following is an extreme instance of the unreliability of tables:

$$\begin{aligned} \log 9\cdot 321 &= 0\cdot 9694, \\ \text{antilog } 0\cdot 9694 &= 0\cdot 919. \end{aligned}$$

F. C. Boon.

612. [K¹. 23. a.] *Conical Projection of Circle into Ellipse.*

The following remarks may be of interest to the engineering draughtsman.

Under conditions, some of which will be stated later in square brackets. the conical projection of a circle is an ellipse. It is required to find the points on the circle that project into the extremities of the principal (major and minor) axes of the ellipse.

The circle $ABCD$ to be projected is supposed to be in the plane of paper of Fig. Let G be its centre and r its radius. Let MN be the line in which

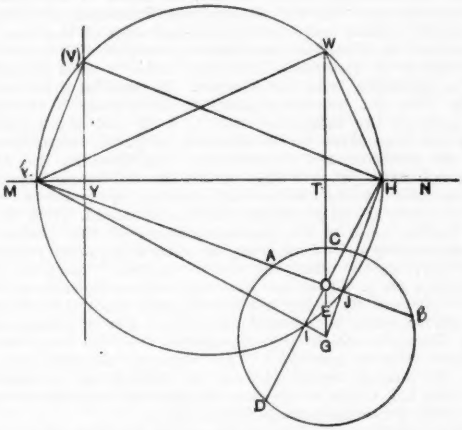
the plane (Plane α) that is parallel to plane of projection, and that contains V, the vertex, cuts plane of paper.

[(1) Plane of projection not parallel to plane of circle.]

[(2) Perp. distance of G from $MN > r$.]

The line $(V)Y$ is supposed to be perp. to MN. Call that plane that is perp. to plane of paper and cuts it in line $(V)Y$, Plane β. The V belonging to this figure, if we restrict ourselves to keeping it outside the plane of the paper, may be *any* point except two on the circle (the V-circle) that lies in Plane β, has Y as centre, and a radius equal to $Y(V)$.

[(3) G not in Plane β.]



Construction.—From G draw GT perp. to MN. On GT take O so that $GO \,.\, GT = r^2$. On GT produced, take on opposite sides of MN, points W and E so that $TW^2 = TE^2 = OT \,.\, GT$. (We can show that one point, say E, will be inside, and the other outside, the circle $ABCD$.) Through E, W, and (V) describe a circle cutting MN at F and H.

From F draw a line through O cutting circle $ABCD$ at A and B.

" H " " " " C and D.

A, B, C, and D are the required four points.

Proof.—Denote projections of A, B, etc., by A', B', etc. Join GF, cutting CD at I. Join GH, cutting AB at J. The V-circle is the intersection of Plane β with a sphere described with centre and radius of circle $EW(V)$. Hence angle $FVH =$ angle $F(V)H$, whatever be the permitted position of V on V-circle; $\therefore$ ang. $F'O'H' = FVH = F(V)H = 90°$.

O being the pole of MN, O' will be centre of ellipse.

But ang. $FWH = 90°$. Thus we have $FT \,.\, TH = TW^2 = OT \,.\, GT$. It readily follows that AB and CD are conjugate with respect to circle. Hence $A'B'$ and $C'D'$ are conjugate with respect to ellipse.

Hence AB and CD project into conjugate diameters at right angles, *i.e.* into the principal axes.

Among the determinations readily obtained, without projecting a point, by means of this theorem are (1) points in the plane of circle that project into foci, (2) ratio and direction of principal axes.

HOWARD E. GIRDLESTONE, M.R.C.S., L.R.C.P.

REVIEWS.

A History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse. By F. Cajori. Pp. 293 + viii (contents), + 6 (index). 7s. 6d. 1920. (Open Court.)

This little book is recommended to all who are interested in the fundamentals of the Calculus: not the least interesting parts of it are the reasoned summaries with which the author closes each chapter under the heading, "Remarks." It is with these remarks that I shall deal.

Chapter I. gives extracts from the writings of Newton, in which he explained, either theoretically or practically, the ideas of fluxions. On p. 32 the author remarks that "at first Newton used infinitesimals (infinitely small quantities) as did Leibniz and other mathematicians of that age"; and on p. 33 he says that, in 1704, the *Quadratura Curvarum* "indicates the almost complete exclusion of quantities infinitely little"; and he endorses this assertion by a quotation from De Morgan. A page later he contends that "not even in 1704 did Newton succeed in completely banishing from his doctrine of fluxions the infinitely little." I do not think that sufficient consideration has been given to the abstract on p. 27, taken from Newton's *Account* of the *Commercium Epistolicum*, published in the *Philosophical Transactions*, vol. xxix. pp. 173-224, where the difference between fluxions as used for rigorous demonstrations and practical applications, which above I have called theoretical and practical, is distinctly stated. With the exception, perhaps, of Wallis, none of the mathematicians of this period considered infinitely small as being an absolute zero, but only as a quantity that was diminished indefinitely, exactly what the word "infinite" implies—without any bounds. Further, on p. 36 we see that the author depends on the English translations of Newton by other mathematicians who understood Newton's practice, but did not really understand his theory. For he quotes a passage, as translated by Thorp, in which occur the words, "at the very *instant* when it arrives." Upon this he remarks: "If 'instant,' as used here, is not an infinitesimal, the passage would seem to be difficult or impossible of interpretation." But the Latin of Newton contains no equivalent to the word "instant." I will quote the passage in full later.

It is probable that Newton had at the outset the idea of an infinitesimal, led thereto by Barrow's teaching of his a-and-e method; and it is likely enough that in Newton's contention, that there is no need of "figuras infinite parvas," we have a distinct allusion to the differential triangle of Barrow, in which it is impossible that infinitely small can have been taken as absolute zero. But quite early, I take it, probably while he was absent from Cambridge for fear of the plague, Newton developed his ideas into a practical instrument, intended merely as such, for use in the problems on motion that occur in the *De Motu*; thus time is made the independent variable. This coloured his rigorous demonstration later when he had to publish his theory in order to claim priority. But in such rigorous demonstration his letter o is never an infinitesimal, but is used to stand for (1) a *finite* quantity, (2) a *variable*, (3) a *unit* of time. Since his other variables did not vary uniformly as a rule, this unit was made indefinitely small, in order that the average velocity over a small interval of time might be considered as a constant velocity over that interval. I use the word velocity designedly; for, with Newton, fluxions *were* velocities, *i.e. time-rates of change*. Thus, if x was a variable, $\dot{x}$ was its "velocity," and $\dot{x}.o$ was its "displacement." Hence Newton is perfectly justified in dividing his equation throughout by o, which is a finite quantity. From this idea, the origin of the use of the word "moment" is apparent; the original Latin, *movimentum*, signified in the first place a slight displacement in a balance, or other state of equilibrium, and later the small additional cause of the slight displacement. Hence $\dot{x}.o$ is the moment of the quantity x; a moment of time is the small increase of time, o, which is to be used as the unit; and the moment of a weight (*vis descensiva*), as used in the eighteenth century, comes from another extension of meaning, and leads, through the

consideration of the *vis descensiva*, at the end of the arm of a lever, to the idea of the moment of a force, as it is understood nowadays.

Now let us consider the passage mentioned above. "Objectio est, quod quantitatum evanescentium nulla sit ultima proportio; quippe quae, antequam evanuerunt, non est ultima, ubi evanuerunt, nulla est. Sed et eodem argumento aeque contendi posset nullam esse corporis ad certum locum pergentis* velocitatem ultimam. Hanc enim, antequam corpus attingit locum, non esse ultimam, ubi attingit, nullam esse." In this first part, the author admits that Newton perceives the difficulty of what we should nowadays call the indeterminate form, 0/0. But Prof. Cajori is not satisfied with Newton's reply. He says: "How does Newton meet this, his own unanswerable argument? He does so simply by stating the difficulty in another form." Now Newton's reply is as follows: "Et responsio facilis est" (note the contrast of the two words, the author's *unanswerable* and Newton's *facilis*). "Per velocitatem ultimam intelligi eam, qua corpus movetur; *neque antequam attingit locum ultimum et motus cessat, neque postea, sed tunc cum attingit.*" The last part, which I have put in italics, was translated by Thorp as follows: "Neither before it arrives at its last place, when the motion ceases, nor after; but at the very instant when it arrives." But the words "tunc cum" should have been rendered by some such logical phrase as "at the *then* when," and not by an equivalent periphrasis introducing infinitesimal notions which are not in the original. The mistake made by Thorp in 1777 is the mistake made by all the interpreters of Newton before and, even in many cases, after that date. If the words "motus cessat" refer to the fact that when the time is absolutely zero there can be no *displacement* (*motus* = movement, in contrast to *motio* = motion); if we admit, with Prof. Cajori, that Newton was aware of the logical difficulty, and was here, in 1687, trying to explain that this difficulty did not exist in the *theory* of fluxions as distinguished from the practice (note the words, "Objectio est"); if we take into account Newton's statement in the introduction to the *Quadratura Curvarum* of 1704: "I fell by degrees upon the Method of Fluxions, which I have made use of here, . . . in the years 1665 and 1666," quoted by the author on p. 21; and if finally we consider the exact meaning and intention of the passage mistranslated by Thorp and italicised above; then, it seems to me that there is no difficulty at all in the interpretation of Newton's meaning. That Newton's explanation does not here explain may be due to difficulties of vocabulary for these new ideas; but it is more probable that Newton did not wish to explain too fully, and was finally forced to do so by what he had heard of the development of the ideas of Leibniz on the Continent. To settle the question is a matter for the very few who have the following qualifications: (i) an expert knowledge of the whole of Newton's works; (ii) a thorough knowledge of the Latin of Newton's time; (iii) a mind unbiassed, because they have read none of the explanations of what fluxions meant, that were given by the followers of Newton. To such, if there are any, I leave the discussion of the following thesis.

1. The definition of a fluxion as given by Newton is logically sound; for a fluxion is defined by a *Schnitt*.

2. There is in it no idea of an infinitesimal.

My arguments are as follows. The letter o, in the *theory* of fluxions according to Newton, is never infinitely small, but is a finite quantity, positive or negative, that may be diminished indefinitely. With these, in theory, there are built up two sequences of *average* velocities, or time-rates of change, viz. a_1, a_2, a_3, ... and b_1, b_2, b_3, ...; each of these sequences has no last term, since the interval of time for which they are the average velocities can be diminished indefinitely. The first set corresponds to Newton's *antequam*, and the second to his *postea*. To fix our ideas, let us suppose that the graph of the fluent on a time axis would be a curve convex to the time axis; then the first series is an ascending series, and the second is a descending series. Now Newton says there is a ratio, which does not belong to either of these series (*neque antequam—neque postea*); in our special case, this ratio is greater than any term in the first series, and less than any term in the second series; all the

* Different readings in the second and third editions.

terms in the first series can be made to differ from a certain quantity by something that is less than any assigned difference by taking o small enough; similarly for the second series; the certain quantity is the same in both cases; and thus the ultimate ratio is defined. Symbolically we have

$$\overbrace{a_1,\ a_2,\ a_3,\ \ldots}^{\textit{antequam}}\quad \overset{\textit{tunc}}{|V|}\quad \overbrace{\ldots\ b_3,\ b_2,\ b_1.}^{\textit{postea}}$$

So much for the theoretical explanation, as given by Newton succinctly but distinctly in the words quoted above. Now let us see if even his *practice* is not just as free from infinitesimal ideas. I suggest that Newton follows the lines of our present practice; that he sees that in most cases he can do without the series equivalent to *antequam*; he sees that the series equivalent to the *postea* leads him, *no matter in what way he continually diminishes his interval of time*, closer and closer to a certain quantity which is the same for all ways. In modern phraseology, all *postea* sequences tend to the same limit. It comes to the same thing ("eodem recidit") if you simply reject the terms containing o, after division by o, in comparison with the terms that do not contain o; or, in practice, for brevity, if you do not write down the terms that would contain squares of o. Similarly, if we are using moments, a, b, c, etc., it is necessary, *in practice*, to retain only the first powers of these. The strongest argument I can bring forward is the method of proof given by Newton for the increment of the rectangle $A \, . \, B$, where A and B are the lengths of the sides which contain it. This was subjected to the most vigorous attack by Berkeley. Prof. Cajori is surprised (p. 91) that Jurin should fail to see the validity of Berkeley's criticism, but the proof is essentially sound. This is seen immediately, if we remember that the moment of a quantity is the product of its fluxion and the small interval of time, o; and secondly, that with three equidistant ordinates on a graph, the chord through the ends of the extremes is more nearly parallel to the tangent at the end of the middle one than it is to the tangent at the end of the first; that is, a better approximation to the velocity is obtained by taking the average velocity over an interval of time with the *then* half-way along the interval than by taking the *then* either at the beginning of the interval or at the end of the interval. Weissenborn's contention that, with equal justice, one might take it two-thirds of the way along the interval is unanswerable; but so also is the reverse argument, that one can take it at the middle if it suits the purpose better. Here it does suit the purpose better; for not only, as stated above, do we get closer approximations for our average velocities, but in this one particular instance, the limit of the sequence of average velocities is apparent at a glance; for *every* term of the sequence is the *same*. Thus, let A and B be two quantities, and a and b their moments; *i.e.* let $a = \dot{A} \, . \, o$, $b = \dot{B} \, . \, o$. Then for any value of o the approximate (average) velocity is

$$\frac{(A + \tfrac{1}{2}\dot{A}o)(B + \tfrac{1}{2}\dot{B}o) - (A - \tfrac{1}{2}\dot{A}o)(B - \tfrac{1}{2}\dot{B}o)}{o}.$$

As o diminishes, this fraction gives a sequence of average velocities having as their limit the fluxion of AB. But for every value of o, the term of the sequence corresponding to this value of o is $A\dot{B} + B\dot{A}$; hence the limit which this sequence defines is also $A\dot{B} + B\dot{A}$. Multiplying by o, we have as the moment of AB the quantity $A\dot{B}o + B\dot{A}o = Ab + Ba$.

It may be argued that in this I am putting forward what I think Newton *might have said* in order to make his proofs rigorous; but I hold that Newton actually intended the meaning that I have given. If this thesis is maintainable, then Prof. Cajori's book is not the History of Fluxions but of Pseudo-Fluxions; it is due to the consideration of the author's argument on p. 36, and to the fact that I wondered what Latin word Newton had used for "instant," that I have hereby directly contradicted an assertion made only a short while ago by myself, that Newton's Fluxions were essentially founded on infinitesimals. I, like Prof. Cajori, trusted to the interpretations

given to Newton's ideas by those who followed him and took part in the controversy with Berkeley; to these, as the author remarks at the end of Chapter II., the significance of Newton's invention was a dead-letter. "What an opportunity did this medley of untenable philosophical doctrine present to a close reasoner and skilful debater like Berkeley!"

Berkeley's remarks on the fluxion of x^n, as quoted on p. 60, show that he has no conception of Newton's view; for he considers that the ultimate value of $(x+o)^n$, when o vanishes, as *the last term* of the series obtained when o is a finite quantity continually and indefinitely diminished, and to this series there is of course no last term. So that Berkeley's arguments are really levelled against the ideas of Newton's mistaken followers, and not against those of Newton—against what I have called pseudo-fluxions, and not against fluxions. Jurin's reply distinctly makes this point, counselling Berkeley to be careful of Newton's words. But Berkeley, in his reply, turns the table very neatly by a dexterous twist of Newton's words (or rather of Ditton's explanation of fluxions as a velocity instead of a time-rate-of-change). Berkeley's whole argument against Newton breaks down, if we allow that the latter did not consider the fluxion as being the last term of the series which defined it. Against the mistaken followers of Newton, Berkeley's argument was unanswerable.

Jurin's defence of the strict Newtonian method, introducing the idea of "not before, nor after," would have been just as unanswerable, if he had not introduced the words "but at the instant when" (p. 81). But, as Prof. Cajori remarks, there is an element of shiftiness about Jurin; and this I put down to Jurin's not really understanding the exact significance of the word *tunc*. It would have been interesting if Berkeley had asked Jurin to justify Newton's method for the moment of $A.B$, by applying the same method to $A.B.C$, or to A^3, as suggested by Sir W. R. Hamilton to De Morgan.*

In the next chapter the author deals with the controversy between Jurin and Robins; and remarks that Robins, in endeavouring to give a clear explanation of Newton's ideas, does not perceive that the rapidity of approach to the limit can be altered in such a way that the limit is reached. But can it be reached without admitting an infinite celerity of approach, which would have been even more objectionable to Berkeley than an infinitesimal? A man, without ever having heard of Cantor, may persuade himself that he has some sort of notion of the infinite and the infinitely small *quantity*; but he never could think of an infinite velocity, which would demand that two distinct positions could be simultaneously occupied; or, geometrically, that a figure could be simultaneously a polygon and a circle, which is impossible because the definition of a circle states that all points on the perimeter of a circle are equidistant from a centre, and this is not the case for a polygon. It seems to me that the author's suggestion amounts to this: that, although it is plain that the circle is not a term in a series of n-gons itself, yet it coincides with the last term of this series. I maintain that here an unfortunate example has been chosen, one of those cases in which the limit is most certainly never attained. The remarks which close this chapter give an accurate summary of the controversy; and the important fact is pointed out that the controversy between Jurin and Robins and Pemberton brought forth valuable constructive results.

The next chapters are bibliographical, Maclaurin's *Treatise* deservedly having a chapter to itself. Of the others who entered the field, none seem to be of much account (p. 206), except perhaps William Emerson and Thomas Simpson; and these gave rise to a fresh controversy amongst their partizans, equally bitter and futile. In this chapter, Prof. Cajori has done exceedingly well to include a quotation (p. 222), which throws into bold relief the popular impression about fluxions—confusion, terminating in disbelief—and the specialist's view that Maclaurin's *Treatise* had made the whole thing clear. The next chapter deals with abortive attempts at arithmetisation. Here I should have expected to find some reference

* Graves, *Life of Sir W. R. Hamilton*, iii. p. 570.

to a little pamphlet by Emerson, called *Arithmetic of the Infinities*,* issued, I believe, with a treatise on Conics,† and that some reference would be given to Wallis's work of the same title. In the next two chapters we have notices of later books and articles on fluxions, and criticisms by British writers; and so we come to Woodhouse (1803), Hales (1804), and the translation by Herschel, Babbage and Peacock of the *Treatise on the Differential and Integral Calculus* by Lacroix. With regard to this, I cannot accept the statement that "Lacroix's definition does not prohibit the limit to be reached" (p. 271). I have before me a copy of this translation, and on p. 5 I read: "We must take care not to confound the differential with the difference $u'-u$." The difference can be made to approach as near to the differential as we choose to make it, but can never equal it in the face of this stipulation. Again, on p. 272, the author contests the statement of the translators that the ancients considered a limit as "some fixed quantity, to which another of variable magnitude can never become equal, though in the course of its variation it may approach nearer to it than any difference that can be assigned." The translators are right; and it was due to this that, after using the method of limits to point out the truth, they turned to the method of exhaustion to give what appeared to them to be a proof with that rigour which could not in their opinion be given by the method of limits. And why not? Because this limit was never reached. We have an example of this in Archimedes' first proof of the centre of gravity of a triangle, which could have been completed directly by the method of limits, instead of which Archimedes uses an indirect method on somewhat similar lines; and then, not being satisfied with it, proceeds to give another indirect proof which does not contain the infinitesimal idea, but depends on the homothetic position of centroids of similar figures, which he assumes as an axiom.

The last chapter of all gives a summary of the merits and defects of the fluxional concepts. This is extremely interesting and valuable, though the author rather inclines to hold cheaply the practical mathematician whose sole purpose is to get things done and who is content to allow the theoretical mathematician all the credit and the pleasure of telling him how to do it. The former, as a class, is as worthy of honour as the latter. I have no doubt there are many experts in mathematical electricity who could not start up an electric motor of commercial size without blowing out or burning some of its parts. Who shall distinguish the respective places of a Kelvin and an Edison? As an example in favour of the non-logical mathematician, let us take the point mentioned on p. 219, the value of x^x when $x=0$. This is a matter for the experts among theoretical mathematicians. The practical man graphs $y=a^x$ for positive values of a and obtains a series of curves, *all* of which pass through (0, 1). These curves, as a diminishes, get closer and closer to Ox on the positive side, and to Oy on the negative side; the limiting value to him is *evidently* the pair of lines Ox, Oy; hence, $a^0=1$ for all values of a, except when $a=0$; $0^x=0$ for all values of x except $x=0$; and 0^0 is indefinite. Who shall blame him if he says: "I don't care what 0^0 is; I shall never have to consider what a^x may be, when *both* of them are zero; so what's the use of troubling about it?"

The book deserves very careful reading, and there is an excellent index. Would that every scientific text were so excellently furnished!

J. M. CHILD.

---

99. Divide my truest kindness among all the dear Newtonians.—*Mrs. Thrale* to Fanny Burney, July 19, 1780. [The Burneys lived in Newton's house in St. Martin's Street.]

100. Major Temple, before leaving his house, would place his four sons facing the wall in opposite corners of a room to solve arithmetical problems correctly before his return.—*N. & Q.* IX. xi. 261. [This was the father of Archbishop Temple.]

---

*[8vo, 1767]. †[pp. 225].

CORRESPONDENCE.

DEAR SIR,

It has been suggested to me that I was unjust to Mr. Jones in my review (vol. x. p. 321) of his useful *First Course of Statistics* when I raised objection to a remark on birth and death rates. His statement is correct as it stands if he includes, as he very probably does, in the "law of increase" of "capital accumulating at compound interest" the cases where there is no increase or a decrease, and if he assumes the conditions have been continuous in the past to reach constancy. I apologise to him if, as I fear, I have been unfair, and to readers of the *Gazette* for my inefficiency.—Yours truly,

W. P. ELDERTON.

Nov. 1921.

YORKSHIRE BRANCH.

THE Annual Meeting of the Branch was held on 3rd December at University House, Leeds. Professor W. P. Milne of Leeds University, Rev. A. V. Billen of Leeds Grammar School, and Mrs. Pochin, Thoresby High School, Leeds, were re-elected President, Secretary and Treasurer respectively for another year; and in place of Miss Greene, Miss Cull and Mr. Blacklock, who retire from the Executive Committee this year, there were elected Miss Stephen of the Girls' High School, Leeds, Miss Sykes of Chapel Allerton High School, Leeds, and Dr. Brodetsky of Leeds University.

An address was given by Dr. Cargill Knott, Reader in Applied Mathematics in the University of Edinburgh, on the life and work of Professor, P. G. Tait, to whom Dr. Knott was formerly assistant. Dr. Knott commenced with an account of the circumstances which led to the selection of Mr. Tait as Professor at Edinburgh, in which position he proved himself one of the clearest and best of lecturers. He also showed how Professor Tait's fondness for golf suggested to him many problems and led to new discoveries in the dynamics of projectiles; and though the professor's theories and his amateur efforts at the construction of newer types of golf clubs did not recommend themselves to the professionals, yet their truth is now universally recognised and their principles applied in present-day clubs. The account of the visit of Helmholtz to Professor Tait led Dr. Knott to describe many of the personal habits and character of the professor, of whom we gained a new view which could never have been gained from his books and which perhaps could have been given us only by one who knew him as intimately as did Dr. Knott. The account of his controversies with Tyndall led Dr. Knott to quote some of the verses of Maxwell, who, like Lord Kelvin, was a close friend and collaborator of Professor Tait. The address was described by Professor Milne, in proposing a vote of thanks as the best lecture of the kind without exception which he had listened to. A most interesting and instructive meeting concluded with tea.

THE PILLORY.

A piece of work can be done by A and B in 4 days, by A and C in 6 days, and by B and C in 12 days; find in what time the work can be done by each working separately. (Middle Grade Algebra Paper, 1921, Intermediate Board for Ireland.)

Two other questions in this paper contained misprints.

S. J. N. MACKINLAY.

ERRATUM.

P. 379, l. 15, *for* "rationelles" *read* "rationnelles."

FOR SALE.

Euler's *Algebra*, translated by Hewlett. 3rd ed. 1822. 4s.

Proceedings of the Congress of Mathematicians at Cambridge, 1912. In excellent condition. 12s. 6d.

Thomson and Tait, *Elements of Natural Philosophy*, 1866-1883. 15s.

De Jonquières, *Mélanges de Géométrie pure*, 1856. 6s.

Apply Box 1, care of Editor, *Math. Gazette.*

THE LIBRARY.

THE Library is at 9 Brunswick Square, W.C., the premises of the Teachers' Guild.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

SCARCE BACK NUMBERS.

Reserves are kept of A.I.G.T. Reports and Gazettes, and, from time to time, orders come for sets of these. We are now unable to fulfil such orders for want of certain back numbers, which the Librarian will be glad to buy from any member who can spare them, or to exchange other back numbers for them :

Gazette No. 8 (very important).
A.I.G.T. Report No. 11 (very important).
A.I.G.T. Reports, Nos. 10, 12.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW

BOOKS RECEIVED, CONTENTS OF JOURNALS, ETC.

January, 1922.

Memoir on Some Properties of Indeterminate Equations. By A. FARKAS. Pp. 9. 1920. (Aiud, Nagyenyed.)

Die Auflösung einiger unbestimmten Gleichungen höheren Grades durch Reihen. By A. FARKAS. Pp. 7. 1920. (Aiud, Nagyenyed.)

A Three-Term Course in Elementary Science. By H. MONTEITH. In 3 parts. Pp. 40 each. 1s. each net. 1921. (Blackie.)

Textile Machine Drawing. By T. WOODHOUSE and A. BRAND. Pp. 124. 2s. 6d. 1921. (Blackie.)

Elementary Algebra. By C. O. TUCKEY. Pp. x+277+xix+xii. 6s. 6d. 1921. (Arnold.)

Méthodes pour résoudre les Problèmes de Géométrie. By J. POIRÉE. Pp. 48. 1921. n.p. (Cocharaux, Auch.)

Useful Engineers' Constants for the Slide Rule, and how they are obtained. By J. A. BURNS. Pp. 78. 3rd edition. 2s. net. 1921. (Percival Marshall.)

Exponentials Made Easy. By M. E. J. GHEURY DE BRAY. Pp. x+253. 4s. 6d. net. 1921. (Macmillan.)

Practical Mathematics for Central and Continuation Schools. By G. SIMMONDS. Pp. 212. 4s. 6d. 1921. (Methuen.)

Œuvres de G. H. Halphen. T. III. Pp. xii-518. 90 fr. 1921. (Gauthier-Villars.)

The Solution of two Famous Problems: How to find the Square of a Circle and Trisect an Angle. By the Rev. J. O'CALLAGAN, P.P. Pp. 22. 5s. net. n.d. (Educational Co. of Ireland, 89 Talbot Street, Dublin.)

History of Symbols for n-Factorial. By F. CAJORI. Reprint from *Isis.* Pp. 414-418. (Summer, 1921.)

Did Fermat have a Solution of the so-called Pellian Equation? By J. M. CHILD. Pp. 255-262. Reprint from *Isis.* (Autumn, 1920.)

Plane Geometry: Practical and Theoretical (Pari Passu). By V. LE NEVE FOSTER. I. Pp. xii+227+viii. II. Pp. 228-422+vii. 3s. 6d. each. 1921. (Bell & Sons.)

Optical Theories. By D. N. MALIK. 2nd ed., revised. Pp. 202. 16s. net. 1921. (Cam. Univ. Press.)

A Treatise on the Integral Calculus, with Applications, Examples, and Problems. By J. EDWARDS. Pp. 907. 50s. net. 1921. (Macmillan.)

Norsk Matematisk Forenings Skrifter. Series 1. No. 5. *On Correlation.* By ALF GULDBERG. Pp. 23. n.p. 1921. (Grøndahl, Christiania.)

Plane Geometry for Schools. By T. A. BECKETT and F. E. ROBINSON. Pp. vii +239. 5s. 1921. (Rivington.)

Newton's Solution of Numerical Equations by Means of Slide Rules. By F. CAJORI. Pp. 245-253. Colorado College Publications. Engineering Series I. No. 18. Nov. 1917.

On Non-Ruled Octic Surfaces whose Plane Sections are Elliptic. By C. H. SISAM. Pp. 640-655. Colorado College Publications. Science Series. XII. No. 15. Nov. 1919.

Swiss Geodesy and the U.S. Coast Survey. By F. CAJORI. Pp. 117-121. Reprint from *The Scientific Monthly.* Aug. 1921.

Elementary Statics of Two and Three Dimensions. By R. J. A. BARNARD. Pp. 254. 7s. 6d. net. 1921. (Macmillan.)

The Rudiments of Relativity. Lectures delivered under the Auspices of the University College, Johannesburg, Scientific Society. By J. P. DALTON. Pp. 106. 5s. 1921. (Wheldon & Wesley, Essex St., Strand.)

Euclid of Alexandria and the Bust of Euclid of Megara. Pp. 141-415. By Florian Cajori. Reprint from *Science*, April 29, 1921.

Suggestions for Students of Mathematics : Mathematics and Life Activities. Pp. 6. (Bulletin of the Department of Mathematics, Brown University, Rhode Island. Number One.)

Specimen Answers of College Candidates in Plane Geometry. Pp. 22. 25 c. Document No. 99. College Entrance Examination Board, New York.

Elementary Physics (First Year). By W. Cameron. Pp. 90. 3s. net. 1921. (Blackie.)

Examples in Optics. Compiled by T. J. I'A. Bromwich. Pp. 16. 2s. net. 1921. (Bowes & Bowes.)

The Fourth Dimension. By E. H. Neville. Pp. 55. 5s. net. 1921. (Camb. Univ. Press.)

Archivio di Storia della Scienza (Nardecchia, Rome). June 1921. *Philip E. B. Jourdain.* Pp. 167-184. By Gino Loria.

A Study of Mathematical Education including the Teaching of Arithmetic. By Benchara Branford. New Edition. Pp. 432. 7s. 6d. net. 1921. (Clarendon Press.)

A History of Greek Mathematics. By Sir Thomas Heath. Vol. I. *Thales to Euclid.* Pp. xv+446. Vol. II. *Aristarchus to Diophantus.* Pp. xi+586. 50s. net. 1921. (Clarendon Press.)

Essai Philosophique sur les Probabilités. By P. S. Laplace. I. Pp. xii+103. II. Pp. ix+108. 3 frs. net each. *Les maîtres de la Pensée Scientifique. Collection de Mémoires et Ouvrages.* 1921. (Gauthier-Villars.)

Pattern Making. By W. R. Needham. Pp. v+114. 2s. 6d. net. (Blackie.)

Jig and Tool Design. By F. Lord. Pp. 69+Charts A-X. 3s. 6d. net. 1921. (Blackie.)

A Budget of Paradoxes. By A. de Morgan. Edited by D. E. Smith. Second Edition. 2 vols. 30s. net. 1921. (Open Court Co.)

Wightman's Secondary School Mathematical Tables. Edited by F. Sandon. Pp. 96. 6d. 1921. (Wightman & Co.)

Applied Calculus. An Introductory Text-Book. By F. F. P. Bisacre. Pp. xv+446. 10s. 6d. net. 1921. (Blackie.)

Elementary Analysis. By C. M. Jessop. Pp. 175. 6s. 6d. net. 1921. (Cam. Univ. Press.)

New Mathematical Pastimes. By Maj. P. A. MacMahon, R.A. Pp. x, 116. 12s. net. 1921. (Cam. Univ. Press.)

Notes sur l'Equation de Fredholm. By B. Hostinsky. Pp. 14. *Notes sur les Quadriques de Révolution qui passant par des points donnés.* By L. Seifert. Pp. 9. *Les Quadriques de Moutard.* By E. Čech. Pp. 17. [In Czech. But summary added in French.] *Géométrie projective de cinq droites infinément voisines.* By E. Čech. Pp. 37. [As above.] 1921. Publications of the Faculty of Science. Masargk University. (Brno, Kounicova 59.)

Publications du Comité Central de la Commission internationale de l'Enseignement mathématique. Edited by H. Fehr.

The American Mathematical Monthly. (The Mathematical Association, Lancaster, Pa.)

Aug.-Sept. 1921.

New Information respecting Robert Recorde. Pp. 296-300. D. E. Smith. *On a Diophantine Problem.* Pp. 300-303. O. D. Kellogg. [The maximum value of any of the unknowns that can occur in a solution in positive integers of $\Sigma x_n^{-1}=1$ is u_n, where $u_1=1$, $u_{k+1}=u_k(u_k+1)$.]

Among my Autographs. P. 303. D. E. Smith. [Delambre, Voltaire.] *On a Diophantine Equation.* P. 307. H. C. Bradley. [$t^3=x^3+y^3+1$.]

Bulletin of the Calcutta Mathematical Society. (Calcutta Univ. Press.)
Sept. 1921.

Graphic Solutions of Spherical Triangles with Applications to Astronomy. Pp. 59-66. G. H. BRYAN. *On the Theory of the Abnormal Zeeman Effect.* Pp. 67-78. P. DAS. *On the Theory of Continued Fractions.* Pp. 79-84. H. DATTA. *On the Inverse of an Undegenerate Non-Plural Quadrate Slope.* Pp. 85-92. S. DHAR. *Percussion Figures in Isotropic Elastic Solids.* Pp. 93-96. S. BANERJI. *On the Evaluation of some Recurrents and Bigradients and on the Expansion of some Functions as Power Series.* Pp. 97-126. H. DATTA. *A Theorem in Determinants. On Algebraic Remainders. On a Theorem in Simultaneous Equations. On the Solutions of a Set of Simultaneous Equations.* Pp. 127-134. H. DATTA.

Gazeta Matematica. (Göbl, Strada Regală, 19, Bucharest.)
August, 1921.

Două probleme militare (urmare şi sfârşit). M. GHERMANESCU.

Sept. 1921.

Un vechiu manuscript de Arithmetica. Pp. 12-21. C. CLIMESCU. *Asupra conicelor circumscrise sau tritangente unui triunghiu.* Pp. 21-25. C. MATEESCU.

Oct. 1921.

Un vechiu manuscript de Arithmetica. Pp. 41-51. C. CLIMESCU. *Asupra unei clase de identităţi.* Pp. 51-55. N. ABRAMESCU.

Nov. 1921.

Un vechiu manuscript de matematică. Pp. 74-87. C. CLIMESCU.

The Journal of the Indian Mathematical Society. (S. Varadachari, Madras.)

On the Cartesian Oval (2nd Paper). Pp. 121-132. V. RAMASWAMI AIYAR. *On the Expansion of Certain Functions (with Properties of Associated Coefficients).* Pp. 133-146. C. KRISHNAMACHARI. *Linear Systems of the Third Order on a Conic.* Pp. 147-150. R VAIDYANATHASWAMI.

Niew Archief voor Wiskunde. (Deslman en Nolthenius, Amsterdam.)
Tweede Reeks. Deel XIII. Vierde Stuk.

De Hypothese van Bayes en de Foutentheorie. Pp. 342-348. W. P. THIJSEN. *Über das Problem der Rundreisen und einem damit im Zusammenhang stehenden Satz von Tait.* Pp. 348-360. F. FITTING. *Stelsels van 7 vier-deelingen van 8 elementen.* Pp. 361-382. N. R. PEKELHARING. *Ueber die wesentliche Identität der Markoffschen und der Leclert-Catalanschen Methode der Reihentransformation.* Pp. 383-394. H. B. A. BOCKWINKEL. *Ueber das kummersche Konvergenzkriterium.* P. 395. H. B. A. BOCKWINKEL. *Over de invoering van 't begrip kromtestraal.* Pp. 396-398. W. VAN DER WOUDE. *Over de normalen van een oppervlak van den tweeden graad.* Pp. 399-406. W. VAN DER WOUDE.

Periodico di Matematiche. (Zanichelli, Bologna.)

Che cosa contiene la " Géométrie " di Cartesio ? Pp. 313-325. E. BOMPIANI. *La prospettiva e lo sviluppo dell' idea dei punti all' infinito.* Pp. 326-337. U. CASSINA. *Il teorema Descartes-Eulero relativo ai poliedri* $[f+v=s+2]$. Pp. 337-346. A. MARONI. *La questione dei nove valori nella risoluzione della cubica.* Pp. 347-354. G. FURLANI. *Polemica logico-matematica.* Pp. 354-365. C. BURALI-FORTI, F. ENRIQUES.

Proceedings of the Edinburgh Mathematical Society. (Bell & Sons.)
Nov. 1921.

Multiply Perspective Polygons inscribed in a Plane Cubic Curve. Pp. 2-6. D. G. TAYLOR. *An Analytical Treatment of the Cam Problem.* Pp. 7-12. G. D. C. STOKES. *Asymptotic Expressions for the Bessel Functions and the Fourier-Bessel Expansions.* Pp. 13-20. T. M. MACROBERT. *Some Extensions of Pincherle's Polynomials.* Pp. 21-24. P. HUMBERT. *Taylor's Theorem and Bernoulli's Theorem: A Historical Note.* Pp. 25-33. G. A. GIBSON. *The Vibrations of a Particle about a Position of Equilibrium.* Pp. 34-57. B. B. BAKER and E. B. ROSS. *On the Relation between Pincherle's Polynomials and the Hypergeometric Function.* Pp. 58-62. B. B. BAKER. *Addition of a Third of a Period to the Argument of an Elliptic Function.* Pp. 63-68. D. G. TAYLOR. *Mathematical Notes: A Problem in Probability.* Pp. 69-71. W. MILLER. *Analytical Note on Lines forming an Harmonic Pencil.* Pp. 71-2. N. M'ARTHUR. *Geometrical Proofs of Trigonometrical Ratios of the Half-Angles of a Triangle.* Pp. 72-74. A. D. RUSSELL. *Length of Perpendicular from a Point to a Line.* Pp. 75-76. J. M'WHAN. *Formulae for the Construction of Right-Angled Triangles.* Pp. 76-77. A. DANIELL. *Note on Isogonal Conjugates.* Pp. 77-79. R. F. DAVIS.

Proceedings of the Physico-Mathematical Society of Japan. (College of Science, Tôkyô Imperial University.)

April, 1921.

On General Modulus of Analytic Functions. Pp. 48-58. S. KAKEYA.

May, 1921.

Equation of Motion of a String and Membrane as derived " d'Energie d'Accélération."

June, 1921.

On an Indeterminate Equation. Pp. 78-92. T. Takenouchi.

July 1921.

On Algebraic Equations having Roots of Limited Magnitude. Pp. 94-100. S. Kakeya.

Revista de Matemáticas y Físicas elementales. (Buenos Aires, 147 Peru.)

Feb. 1921.

Progresiones Aritméticas y Geométricas. Pp. 217-219. J. S. Corti. *Notas sobre la Determinación elemental del Centro de Curvatura en los Vértices de las Cónicas* (concluded). Pp. 210-215. C. C. Dassen.

March, 1921.

Teoria elemental del péndulo matemático. Pp. 241-249. H. Uhlenburg, A. Riva. *Una nota sobra combinaciones y algunas aplicaciones.* Pp. 249-252. A. E. de Cesaré.

May, 1921.

Dos teoremas de la teoria de los números. I. G. Lintes. Pp. 3-4.

July 1921.

Sobre las coordenadas proyectivas. Pp. 49-56. F. D. Jaime. *Suma de potencias de números naturales.* Pp. 56-61. G. Itzigsohn.

August, 1921.

Notas de geometria analitica. Pp. 73-78. E. Rebuelto. *Sobre las coordenadas proyectivas.* Pp. 78-84. F. D. Jaime.

Oct. 1921.

Cálculo de algunas Integrales Trigonométricas indefinidas. Pp. 121-126. E. Rebuelto. *Sobre el número de Aleaciones metálicas.* Pp. 127-129. J. Babini.

Revista Matemática Hispano-Americana. (Soc. Mat. Espanola, Madrid.)

Jan.-Feb. 1921.

Iniciación en la Mecánica analitica. Pp. 11-16 (conc.). F. L. de Nó.

June, 1921.

Teoria de las Superficies de Gauss. Pp. 167-172. G. Fubini. *Sobre algunos principios de la Teoría de los Conjuntos.* Pp. 173-177. R. M. Aller.

Revue Semestrielle des Publications Mathématiques. (Gauthier Villars.) Tome XXIX. 1920. April-October.

School Science and Mathematics. (Smith & Turton, Mount Morris, Illinois.)

March, 1921.

Mathematical Ability as related to General Intelligence. Pp. 205-215. B. R. Buckingham.

April, 1921.

Relation of Science and Mathematics to Business and Industry. Pp. 327-334. F. D. Barber. *From the Complex to the Simple.* Pp. 343-349. J. A. Nyberg. *A Graphical Solution of Empirical Relations of one Independent Variable in a Function containing four Undetermined Functions.* Pp. 358-365. W. Bartky.

May, 1921.

Teaching Formulae in the Junior High School. Pp. 409-417. J. A. Nyberg. *The Mathematical Association of America.* Pp. 418-422. G. A. Miller. *Diophantine Analysis applied to the Construction of Regular Polygons.* Pp. 422-424. M. O. Tripp.

Oct. 1921.

A Dream came True. Pp. 621-627. W. A. Austin. *The Extension of a Process of Factorisation.* Pp. 628-630. S. M. Karmalkar. *Falling Bodies in Ancient and Modern Times.* Pp. 638-648. F. Cajori. *Gerbert's Letter to Adelbold.* Pp. 649-653. G. A. Miller. *The Mathematics needed in Freshman Chemistry.* Pp. 654-665. L. W. Williams. *Do High School Pupils dislike Mathematics?* Pp. 674-675. G. Gingery.

Nov. 1921.

An Analysis of an Experiment in Teaching First Year Mathematics. Pp. 757-764. I. Holroyd. *Some Plane Geometry Problems.* Pp. 765-769.

Scientia. I.-III. 1921. (Williams & Norgate.)

Les contributions des différents peuples au développement des mathématiques, I^e partie ; *Evénements mémorables et hommes représentatifs dans l'histoire des mathématiques.* Pp. 169-184. Gino Loria.

Sphinx-Oedipe. Aug. 1921. (A. Gérardin, 32 Quai Claude le Lorrain, Nancy. 17 frs. a year.)

THE MATHEMATICAL ASSOCIATION.

(*An Association of Teachers and Students of Elementary Mathematics.*)

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto."—BACON.

THE MATHEMATICAL ASSOCIATION, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and has exerted an important influence on methods of examination.

The Annual Meeting of the Association is held in January. Other Meetings are held when desired. At these Meetings papers on elementary mathematics are read and discussed.

Branches of the Association have been formed in London, Southampton, Bangor, and Sydney (New South Wales). Further information concerning these branches can be obtained from the Honorary Secretaries of the Association.

"The Mathematical Gazette" (published by Messrs. G. BELL & SONS, LTD.) is the organ of the Association. It is issued at least six times a year. The price per copy (to non-members) is usually 2s. 6d. each. The *Gazette* contains—

(1) ARTICLES, mainly on subjects within the scope of elementary mathematics;

(2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books;

(3) REVIEWS, written by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim, where possible, is to dwell on the general development of the subject, as well as upon the part played therein by the book under notice;

(4) SHORT NOTICES of books not specially dealt with in the REVIEWS;

(5) QUERIES AND ANSWERS, on mathematical topics of a general character.

Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 15s. per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette."